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BIOGRAPHY.

DR. JOEL E. HENDRICKS, A. M.

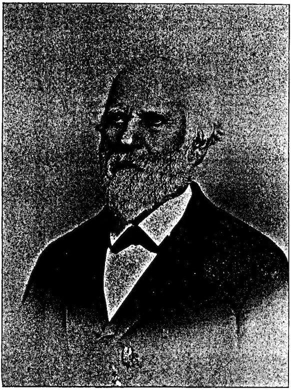
BY J. M. COLAW.

Joel E. Hendricks was born in Bucks county, Pennsylvania, March 10, 1818, and died at his home in Des Moines, Iowa, June 8, 1893.

In 1824, his father moved his family to Columbiana county, Ohio, where the subject of this sketch worked with his father and brothers on a farm until the fall of 1836. Up to this time he had only attended school at intervals, but had learned to read and write and was considered an expert in arithmetic. At the age of eighteen he taught his first school, and the next year (1837), with his father's consent, he bound himself as apprentice to a mill-wright for two years, with the condition that he need not work at his trade during the winter. In 1837-8, he again taught school, and by some means came into possession of a copy of John Hamilton Moore's Navigation and a copy of Ostrander's Astronomy. He soon became deeply interested in these books and devoted all his spare time to their study, with the result that he soon became quite expert in Trigonometry and learned to calculate and project both solar and lunar eclipses with considerable facility.

Young Hendricks spent the winter of 1838-9, working at his trade, in St. Charles county, Missouri, but returned to Ohio the following summer, and again taught school during the winter of 1839-40. During this winter he procured a copy of Bridge's Algebra, upon which he spent two hours each night, solving all the examples the book contained in five weeks.

In the spring of 1840, he visited Abijah McLean, the county surveyor, and a mathematician of good repute, and formed an acquaintance which afterwards ripened into a close friendship. To this friend, he owed much for his mathematical knowledge afterward acquired. From McLean's library he obtained most of the mathematical books which he afterward studied, including Hutton's Mathematics, Newton's Principia, and Bowditch's translation of the Mecanique Celeste.



Dr. Joel E. Hendricks, A. M.

In this same year, he entered a medical office with the view of reading medicine, preparatory to attending a medical school. He soon became deeply interested in his professional studies, and continued his course of reading, supporting himself by teaching in the winter. In the spring of 1842, he went to western Ohio, intending to practice medicine for a time to accumulate funds to enable him to graduate as an M. D. This expectation was never realized. In the spring of 1843, he married and the increasing demands on his time and means prevented him from afterwards attending a medical college. However, he continued the practice of his profession for twenty years with a fair degree of success.

Dr. Hendricks never aspired to political honors but during the latter part of the time he was engaged in the practice of medicine, and after he had abandoned his practice, he was intrusted with several offices, including that of school examiner, county surveyor, county treasurer, county auditor, U. S. deputy surveyor of Colorado, and deputy provost marshal for DeKalb county, Indiana, during the war.

A few years prior to 1860, he co-operated with several prominent citizens of Newville in the establishment of Newville Academy, and was elected teacher of mathematics, which position he resigned two years later. In 1861, he was awarded a contract by the surveyor general of Colorado, to make certain extensive surveys in Colorado, which, after many delays, difficulties, and dangers from the Indians, he completed October 28, 1861, two days before the time of his contract expired.

In the fall of 1864, he removed from DeKalb county, Indiana, to Des Moines, Iowa, where he afterward resided. During the first ten years after he removed to Des Moines, he was mainly engaged in surveying, and during the next ten years, viz.; from 1873 to 1883,he was exclusively engaged in editing and publishing *The Analyst.*

In 1865 he received an honorary A. M. from Indiana State University. In 1880 he was elected a member, and in 1885 a fellow of the American Association for the Advancement of Science and in 1891 he was elected a member of the New York Mathematical Society.

In the life of Dr. Hendricks, the practical and theoretical were happily blended. As a mathematician he was entirely self-taught. It was as editor of *The Analyst* that he obtained his prominence in the mathematical world. *The Analyst* was started in January, 1874, and was conducted with such marked ability, that its reputation became almost world-wide. Much of this deserved success was due to the untiring energy of its editor. Its discontinuance, after an existence of ten years, was not for want of support, but due to the failing health of Dr. Hendricks.

Last year, the State Historical Society of Iowa added to its collection of valuable papers a complete set of the *Analyst*, together with a biographical sketch of the editor, and many letters, which show the high estimation in which Dr. Hendricks was held by the mathematicians of his time. Prof. Pelz, of the Technical High School, of Graz, Austria, wrote to Dr. Hendricks for his portrait,and spoke most flatteringly of the ability displayed in conducting the *Analyst.* Prof. Glaisher, of Trinity College, England, expressed his high appreciation of

it. La Societe Physico-Mathematique, of the Imperial University of Kasan, and other scientific bodies, recognized the standing of the *Analyst* by soliciting an exchange of their publications. Mathematical specialists in Edinburg, Paris, and other centers of learning showered upon the editor of the *Analyst* evidences of their regard. From the colleges and universities of the United States came letters indicating the highest appreciation of Dr. Hendricks' services to science. The *Analyst* was deemed worthy a place in the observatory of Greenwich, England, and the famous astronomer, Schiaparelli, of the Milan observatory, wrote to Dr. Hendricks at length on mathematical subjects.

These references serve to show that the *Analyst* was regarded as a real promoter of mathematical progress and of genuine service to mathematicians. The life of a man capable of achieving such success, from so obscure a beginning and under circumstances so unfavorable, cannot fail to be of interest to the readers of the MONTHLY.

After the discontinuance of the *Analyst*, Dr. Hendricks continued to manifest his interest in mathematical subjects by frequent contributions to other periodicals, and his writing always commanded wide respect.

Though gifted beyond the ordinary, Dr. Hendricks was modest to a fault, and thought little of self or self interest. His life was characterized by candor, modesty, and devotion to the truth.

Dr. Hendricks had been in failing health for some time, but was not considered dangerously ill, until a few hours before his death. He passed away surrounded by his family, which consists of Mrs. Hendricks and six daughters.

A DEFECTIVE PROOF IN SOLID GEOMETRY.

By C. W. M. BLACK, A. M., Professor of Mathematics in the Wilmington Conference Academy, Dover. Delaware.

On one occasion while engaged in an effort to render more evident to my class in Solid Geometry the text-book proof of a theorem which I had not before examined carefully, I became aware of its faulty character. Though the text-book* bears the date 1888, and is extensively used, the proof still remaines uncorrected and, so far as I have observed, unchallenged.

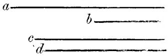
In proving "Two triangular pyramids having equivalent bases and equal altitudes are equivalent,"† we are told to divide the common altitude into a number of equal parts and pass planes through the points of division parallel to the bases, thus making the corresponding sections of the two pyramids equivalent. Then on the base and each section of one pyramid as lower bases, prisms are constructed with lateral edges equal and parallel to the division of

* Wentworth.
† Book VII. Prop. XVI.

one edge of the pyramid, and on the sections of the other as upper bases prisms are constructed similarly.

"The sum of the first series of prisms is greater than $S-ABC$, and the sum of the second series is less than $S'-A'B'C'$; therefore, the difference between $S-ABC$ and $S'-A'B'C'$ is less than the difference between the sums of these two series of prisms."

The last conclusion quoted contains an unwarranted assumption, based on the supposition that if $a>b$ and $c>d$, the difference between a and d is greater than the difference between b and c which is not necessarily true. An illustration by means of lines also will show the fallacy. The error probably arose from overlooking the fact that so far as we know from the conditions assumed, there is no reason to suppose the second pyramid not greater than the first.



This proof, which closes by showing that the difference of the two series of prisms, being equal to the lowest prism of the first series, can be made less than any assignable volume by indefinitely increasing the number of prisms, and that the difference of the pyramids thus cannot be any assignable volume, replaces in an older edition a proof based on the theory of limits. Unless it was to relieve the monotony of proof by limits, it is not easy to see why the change was made.

I have also found the same proof in a recent geometry by another author.‡ In a third§ I find a similar idea, but used in a legitimate way. Here both sets of prisms are constructed on one pyramid and the limit of the sum of the inscribed series is thus shown to be the pyramid. Then by inscribing a series in each of the pyramids with equivalent bases and equal altitude, the method of limits is used to prove the pyramids equal. The other method may be considered an abbreviation of this, but its introduction of the objectionable feature ought to rule it out.

REMARK ON DIVISION OF CONCRETE NUMBER.

By C. H. JUDSON, LL. D., Professor of Mathematics, Furman University, Greenville, South Carolina.

Professor Ellwood's article, in the February Number of the MONTHLY, suggests the question—What is a *Concrete Number?*

Number answers to the question—How many ? Quantity (quantus) answers to the question—How much ?

Twenty gallons is a concrete *expression*, representing *quantity*. It is complex, and consists of one expression which represents—how many, and another which represents a concrete unit of *quantity*. It is, therefore, something more

‡ Stewart, Book IX., Prop. IX.

§ Byerly's Chauvenet, Book VII. Props. XV. and XVI.

than Number; unless we accept the Pythagorean notion—that "*Number* is the essence of *things*."

Most definitions and illustrations of *Concrete Numbers* are ambiguous. Prof. Simon Newcomb (see Alg. p. 3) says, "A concrete number is one in which the kind of *quantity* which it measures" (?) "is expressed or understood; as, Seven *miles*, Three *days*." (*Italics his.*)

Webster says, "Concrete number is a number *associated with*, or *applied to* a particular object; as, *Three* men, *five* days." Which of the two is correct? Do the italicised words of the former or of the latter express *numbers?* If the latter, then the so-called *Denominate Numbers* of our Arithmetics should be "Denominate Quantities."

Now, if we define division as a process of finding one of the equal parts of a *number*, may we not substitute *concrete quantity* for *number?* A fourth part of \$20 is \$5 and four times \$5 are \$20.

We wish to know what is a fourth part of 5768 acres. How shall we find it? What name shall we give to the process? Prof. Ellwood would say "Take a fourth part of the *number* 5768 and annex the name of the concrete unit.

We suggest that a *concrete number* cannot be used as a multiplier, because it is a *quantity* and *not* a *number*.

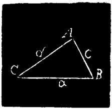
OBLIQUE ANGLED TRIANGLES.

By RALPH H. KUNSTADTER, Graduate Student, Yale University, New Haven, Connecticut.

Solution of an oblique angled triangle, given the two sides and the angle included between them.

The formula found below, was not known to me being in use, and have ascertained its existence only after having derived it originally in a simple way from the fundamental theorem of trigonometry, namely; the proportion of the sines of the angles is direct with that of the sides opposite to them.

Given in an oblique angled triangle for instance, the sides a, b, and angle C, and we wish to find directly from the given quantities the angle B.



$\dfrac{a}{b}=\dfrac{\sin A}{\sin B}\ldots.(1)$. Equation (1) has two unknown quantities, we must therefore get rid of A, in order to solve our example. $\dfrac{a}{b}=\dfrac{\sin[180^\circ-(B+C)]}{\sin B}=\dfrac{\sin(B+C)}{\sin B}\ldots(2)$ $\dfrac{a}{b}=\dfrac{\sin B\cos C+\cos B\sin C}{\sin B}\ldots.(3)=\cos C+\cot B\sin C$ $\ldots(4)$. $\cot B=\dfrac{a}{b\sin C}-\cot C\ldots.(5)$. In order to make equation (5) convenient for the application of logarithmic tables, assume $\dfrac{a}{b\sin C}=\cot\phi$. Hence,

we have $\cot B = \cot\phi - \cot C \ldots (6)$ and also $\cot B = -\dfrac{\sin(\phi - C)}{\sin\phi\sin C} \ldots (7)$ or $\tan B = \dfrac{\sin\phi\sin C}{\sin(C-\phi)} \ldots (8)$.

As to the practical value of this formula, the less we have to open the book of logarithms the better the formula is, and therefore $\tan\dfrac{B-A}{2} = \dfrac{b-a}{b+a}\cot\dfrac{C}{2}$ is preferable to the above, but as regards the theoretical rank, it is perhaps of the same degree.

It is interesting to notice that an important formula can also be obtained with the same processes from the oblique-angled spherical triangle, (but only for limited cases). Given again a, b, C, by drawing a spherical triangle and lettering it the same as in the above figure,

$\dfrac{\sin a}{\sin b} = \dfrac{\sin A}{\sin B} \ldots (\mathrm{I}) \quad = \dfrac{\sin(B+C)}{\sin B} \ldots (\mathrm{II}).\quad \dfrac{\sin a}{\sin b} = \cos C + \cot B\sin C \ldots$ (III), $\cot B = \dfrac{\sin a - \sin b\cos C}{\sin b\sin C} = \dfrac{\sin a}{\sin b\sin C} - \cot C$.

We might again assume $\dfrac{\sin a}{\sin b\sin C} = \cot\phi$, and proceed as above, or find the tangent or cotangent of the sum of the two unknown angles divided by two.

NON-EUCLIDEAN GEOMETRY, HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph.D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

CHAPTER FIRST.

EUCLID.

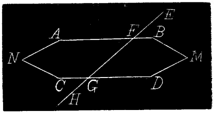
When Alexander the Great was conquering the world, from Macedonia to the Indus, from the Caspian sea to the cataracts of the Nile, founding at least eighteen cities named Alexandria, little did he think that with the only one of these the name now suggests would be connected a man destined to give his name to the universe; for all spaces are now Euclidean or Non-Euclidean.

Euclid, after the death of Alexander, was called by Ptolemy Lagus to open the mathematical school of the first true university, that at Alexandria, and on its teaching so impressed his individuality, that henceforth his name and his immortal Elements stood for his science itself.

Says De Morgan: "As to writing another work on geometry, the middle ages would as soon have thought of composing another New Testament. ....his order of demonstration was thought to be necessary, and founded in the nature of our minds.

The story about Pascal's discovery of geometry in his boyhood (A. D. 1635) contains the statement that he had got 'as far as the 32nd proposition of the first book' before he was detected, the exaggerators (for much exaggerated this very circumstance shows the truth must have been) not having the slightest idea that a new invented system could proceed in any other order than that of Euclid." But even from remote antiquity objection had been made against one point in this imperishable bible of geometry, against Euclid's treatment of parallels.

The great mathematician Ptolemy (died A. D. 168), whose system of astronomy ruled alone until Copernicus, in a work on pure geometry of which Proclus has preserved extracts, discussed the propriety of Euclids famous *Parallel-Postulate*, and proposed to substitute for Euclid's treatment the following: Let the straight $EFGH$ meet the straights AB and CD so as to make the sum of the conjugate angles BFG and FGD equal to two right angles.



It is required to prove that AB and CD are parallel. If possible let them not be parallel, then they will meet when produced say at M (or N). But the angle AFG is the supplement of BFG, and is therefore equal to FGD: similarly the angle FGC is equal to the angle BFG. Hence, the sum of the angles AFG and FGC is equal to two right angles, and the lines BA and DC will therefore if produced meet at N (or M). But two straight lines cannot enclose a space, therefore, AB and CD cannot meet when produced, that is they are parallel.

Inversely, if AB and CD be parallel, then AF and CG are not less parallel than BF and GD; and therefore whatever be the sum of the angles AFG and FGC such must also be the sum of the angles FGD and BFG.

But the sum of the four angles is equal to four right angles, and therefore the sum of the angles BFG and FGD must be equal to two right angles.

Proclus calls this *paralogismos* and *deixeos astheneia;* and Barocius the Venetian translator in 1560, notes it in the margin as *Flagitiosa Ptolemaei ratiocinatio.* "The palpable weakness is, that there is no proof, evidence, or cause of probability assigned, why parallelism should be connected with the angles on one side being together equal to those on the other; the very question in debate being, whether they may not be a little more than two right angles on one side and a little less on the other, and still the straight lines not meet."

The famous Nasir-Eddin (about A. D. 1240), who was born at Tus in Khorassan and says that Euclid was born there also (the Arabs tried to claim oriental origin or education for all the great Greek mathematicians) founds a parallel-theory on two postulates: I. If perpendiculars let fall from one straight onto another make, on the same side, acute angles with the first, these perpendiculars are always shorter toward the acute-angled side, longer toward the other side.

II. Inversely, if the perpendiculars to one straight always grow shorter toward one side, their angles with the other straight are acute on this side.

In a manuscript copy of Euclid in Arabic but in a Persian hand, bought at Ahmedabad, the editor, on the introduction of the Parallel-Postulate, says: "I maintain that the last proposition is not among the universally acknowledged truths, nor anything that is demonstrated in any other part of geometry.

The best way therefore would be that it should be put among the questions instead of the principles; and I shall demonstrate it in a suitable place.

And I lay down for this purpose another proposition, which is, that straight lines in the same plane, if they are subject to an increase of distance on one side, will not be subject to a diminution of distance on that same side, and the contrary; but will cut one another."

Clavius, author of an edition of the Elements: Euclidis elementorum, libri XV.; accessit XVI. de solidorum regularium comparatione, etc. Romae, 1574, announces that "a line every point in which is equally distant from a straight line in the same plane, is a straight line;" upon taking which for granted he infers the properties of parallels. He supports his assumption on the ground that because the acknowledged straight line is one which lies evenly [*ex aequo*] between its extreme points, the other line must do the same, or it would be impossible that it should be everywhere equidistant from the first. He adds: Neque vero cogitatione apprehendi potest aliam lineam praeter rectam, posse habere omnia sua puncta a recta linea, quae in eodem cum illa plano existat, aequaliter distantia. [Clavii Opera. In Euclid. Lib. I. p. 50.]

But it may be remarked, that though the equidistantial be situated *ex aequo* with reference to the primitive straight, we know not whether it posesses this property with reference to itself.

Borelli defines the parallel to a straight by saying that it is the line equidistant from the straight.

In a tract now in the British Museum, printed in 1604, by Dr. Thomas Oliver of Bury, entitled *De rectarum linearum parallelismo et concursu Geometrica*, two demonstrations are proposed; both of them depending on taking for granted, that if a perpendicular of fixed length moves along a straight line, its extremity describes a straight line.

Wolfius, Boscovich, Thomas Simpson, and Bonnycastle define parallels as "straight lines which preserve always the same distance from one another", by distance being understood the length of the perpendicular drawn from a point in one of the straight lines to the other.

But no evidence is adduced that straight lines in any assignable position, *will* always preserve the same distance from one another; nor that the equidistantial is a straight.

Varignon, (1654—1722), the intimate friend of Leibnitz, proposes to define parallels to be "straight lines which are equally inclined to a third straight line," or in other words, make equal corresponding angles. By this he either intends to assume the principal thing at issue, which is whether all straights meet except those making such angles; or he intends to admit none to be parallels except those making equal corresponding angles with some one straight; in which case it must also be assumed, that because they make equal angles with one straight, they shall also do it with any.

AN UNREASONABLE RULE IN SURVEYING.

By Professor P. H. PHILBRICK, M. S., C. E., Lake Charles, Louisiana.

The Government requires Surveyors, on account of the more rapid convergence of the meridians north of a parallel of latitude than south, to locate "correction parallels" every 24 miles north and every 30 miles south of the principal base. Where is the parallel of latitude from which the convergency for 24 miles north is equal to the convergency for 30 miles south? Let $x=$ the co-latitude of the required parallel. The arcs of two parallels included between two meridians are as their radii which are as the sines of their co-latitudes. Also 24 miles $=20'.85=a$ (say) and 30 miles $=26'.07=b$ (say).

Hence, we have $\sin(x+b)-\sin x=\sin x-\sin(x-a)$. Expanding it, $\sin x\,(2-\cos a-\cos b)=\cos x\,(\sin b-\sin a)$.

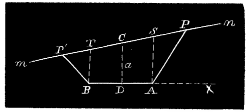
Therefore, $\tan x=\dfrac{\sin b-\sin a}{2-\cos a-\cos b}=\dfrac{\sin b-\sin a}{\text{vers}\,a+\text{vers}\,b}=32.16949$.

Hence, $x=88^\circ\ 13'\ 10''$ and $90^\circ-x=\text{latitude}=1^\circ\ 46'\ 50''$. The rule is not reasonable. The convergency upon each side of all parallels where surveys are made is sensibly the same.

TO SET SLOPE STAKES WHEN THE SURFACE IS STEEP BUT SLOPES UNIFORMLY.

By J. M. BANDY, Professor of Mathematics, Elon College, North Carolina.

Let mn represent the surface of the ground. Let C represent the position of the centre peg, and let CD $(=a)$, the value of which is found from the level notes, represent the centre cut. The width of the road, BA, is b.



It is proposed to find what engineers designate as cuts at P, P', S, and T.

In the direction of P, and in connection with C, one setting of the rod determines the slope of the ground mn. Call this slope m. Now, since the co-ordinates of C are $(o,\ a)$, D being the origin of co-ordinates, the equation of the surface, mn, is $y=mx+a$. . . . . (1).

The quality of the soil determines the slopes of PA and $P'B$. Call this slope m'. Since the co-ordinates of A are $(\frac{b}{2}\ ,\ o)$, the equation of PA is

$y=m'x-\dfrac{m'b}{2}$ . . . . . (2).

Combining (1) and (2), the co-ordinates of P are known. Hence, the

cut at this point is known. Designating the co-ordinates of P, just found, by (x'', y'') and the co-ordinates of C by $(x',y')[=(o, a)]$, $CP=\sqrt{(x''-x')^2+(y''-y')^2}$ ....(3).

Measure this distance, CP, and fix a stake at P.

The equation of SA is $x=\frac{b}{2}$....(4).

Substituting this value of x in (1), the cut at S is at once obtained. Denoting co-ordinates of S by (x'',y'') and co-ordinates of C by (x',y'), and substituting in (3), CS is known. Measure this distance, and fix a stake at S.

The same course of reasoning applies in finding the cuts at P' and T.

The writer has found this method more expeditious than *trial* and *error* when the surface was much inclined. The numerical computations did not require as much labor as maneuvering with the rod and level.

Since he has not seen the above method in any of the books which have fallen under his eyes, he has been induced to give it in the hope that it might prove useful, as well as suggestive, to others under similar circumstances.

NOTE ON MR. ELLWOOD'S REMARKS ON DIVISION.

By DAVID EUGENE SMITH, Ph. D., Professor of Mathematics in the Michigan State Normal School, Ypsilanti, Michigan.

Reluctantly, feeling that it is almost unnecessary, a note is offered on Mr. Ellwood's article on p. 47. Such articles float through the primary journals of education now and then, and possibly do no harm that can be undone by replying to them. But in a mathematical journal such pedagogy should not go unchallenged.

It is a trite idea that all advance in mathematics, and in all science generally, has had to contend with just such arguments as those of Mr. Ellwood. The whole history of the development of mathematics has been a history of the destruction of old definitions, old hobbies, old idols. When Ahmes copied the papyrus which now, after 3500 years, makes him famous, only one fraction could be written whose numerator was not one. A definition or an artificial symbolism created a barrier. For over a thousand years thereafter the Egyptians, Hebrews, Phœnicians, and Greeks recognized that barrier. But finally came an iconoclast and knocked down the sacred wall, and the Greeks learned about fractions.

The Arabs found negative roots to certain quadratic equations. Teachers raised their hands in horror. It was impossible; the sacred definition of number stood in the way. But by and by some one suggested that that definition better stand aside, and so the reality of negative numbers became recognized. Then came some other impossible roots, expressions that involved the square root of a

negative number. Even the serious thinkers among the Arabs rejected them. Cardan in 1545,saw some light on the subject, but even from the mathematicians of his time the definitions of number and square-root barred such seemingly impossible roots. But as this century approached the light grew stronger and finally Gauss and Argand knocked down the barriers and the theory of complex numbers was established.

It was the same with the theory of exponents. 4^2 meant 4 taken twice as a factor. Then some one had the audacity to suggest $4^{\frac{1}{2}}$, and we can fancy the cries that greeted it; and we can almost hear the definition-defenders talking about the absurdity of taking a number one-half of a time as a factor. How they must have writhed when $4^{-\frac{1}{2}}$was first suggested! It was the same old trouble. Somebody saw that a pet definition must be shattered, and amid tears and complaints the fragments were swept out.

It is within our generation that the idol known as the commutative law of multiplication has been placed in the museum of antiquities. One of two things must happen, quaternions must die in the womb, or else the commutative law must yield. It was, too bad, but quaternions conquered. And now the theory of substitutions supports quaternions with such a powerful arm that even primary text-books are properly limiting the application of the law.

Of course the list could be multiplied a thousand fold. But the moral is that the world is going to move along in spite of somebody's definition of fraction, or number, or power, or exponent, or multiplication, or division, and in spite of our veneration for commutative laws, the word "imaginary," such ideas as the primative one of interior angles, and the multifold like.

But Mr. Ellwood will say, that this does not touch his article. On p. 47 he says, "Multiplication is a mere process of adding." Here is the idol, the sacred definition. He might as will have said, "One is not a number," because nobody ever thought of it as such until modern times. So the enlargement of the notion of multiplication is modern. But the enlargement is made, and it will never be unmade. Multiplication was, originally, what Mr. Ellwood says, but the world of to-day says, "*nous avons change tout cela.*" And the world of to-day says $10÷2=$5, and a good part of the world expressly distinguishes between the two notions of division in the primary grades. To try to establish the limitation proposed by Mr. Ellwood is to let the tail try to wag the dog.

ARITHMETIC.

Conducted by B. F. FINKEL, Kidder, Missouri. All contribution to this department should be sent to him.

6. Proposed by B. F. FINKEL, Professor of Mathematics in Kidder Institute, Kidder, Missouri.

What is the volume of a regular pantagonal pyramid, each side of whose base is 10 feet and the altitude 20 feet?

I. Solution by P. S. BERG, Apple Creek, Ohio; J. A. CALDERHEAD, Limaville, Ohio; Professor H. C. WHITAKER, Philadelphia, Pennsylvania; and H. W. HOLYCROSS, Pottersburg, Ohio.

Area of base$=\text{side}^2\times\frac{5}{4}\sqrt{1+\frac{2}{5}\sqrt{5}}$.

Area of base$=10^2\times1.7204774=172.04774$ sq. ft.

$\therefore$ Volume of pyramid$=\dfrac{172.04774\times20}{3}=1146.985$ cu. ft.

II. Solution by F. A. SWANGER, Professor of Mathematics in the State Normal School, Kirksville, Missouri.

$r=\frac{1}{4}\left(\frac{a}{2}(\sqrt{5}+1)\sqrt{10+2\sqrt{5}}\right)$ where r is apothem and a the side of a regular pentagon. Reducing this expression, the constant ratio between the apothem and side of a regular 5-side may be found to be $.6882=\frac{r}{a}$; $r=a\times.6882$. Whence in above problem $r=10\times.6882=6.882$. Now area of a regular n-side=the product of its apothem (r) and $\frac{1}{2}$ its perimeter.

$\therefore$ Base of pentagon$=25\times6.882=172.050$. But volume of a pyramid equals $\frac{1}{3}$ $B\times H$ (altitude).

$\therefore$ $V=\frac{20}{3}\times172.05=20\times57.35=1147.00$.

Neatly solved by *A. L. Foote*, *G. B. M. Zerr*, and *I. L. Beverage*.

7. If an article had cost me 10% less, the gain would have been 12% more; what was the gain per cent?—[Selected from *Brooks' Higher Arithmetic.*]

Solution by B. F. FINKEL, Professor of Mathematics in Kidder Institute, Kidder, Missouri.

II.

1. *100%* = selling price, and
2. 100% = actual cost. Then
3. *100%* − 100% = actual gain.
4. 100% − 10% = 90% = supposed cost. Then
5. *100%* − 90% = gain on the condition that the article cost 90%.
6. 1. 90% = 100% of itself.
   2. 1% $=\frac{1}{90}$ of 100%$=1\frac{1}{9}$%, and
   3. *100%*−90%=(*100*−90) times $1\frac{1}{9}$%$=\frac{10}{9}\times$*100%*−100%=conditional gain%.
7. $\frac{10}{9}\times$*100%*−100%−(*100%*−100%)$=\frac{1}{9}\times$*100%*=difference.
8. 12%=difference.
9. $\therefore$ $\frac{1}{9}\times$*100%*=12%.
10. *100%*=9×12%=108%, the selling price in terms of the cost price.
11. $\therefore$ 108%−100%=8%=the gain.

III. $\therefore$ The gain is 8%.

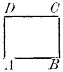
Also solved by *John T. Fairchild*, *G. B. M. Zerr*, *A. L. Foote*, *H. C. Whitaker*, *H. W. Holycross*, *I. L. Beverage*.

8. Proposed by EARL D. WEST, West Middleburg, Logan County, Ohio.

The number of men in a side rank of a solid body of militia is to the number of men in the front rank as 2 is 3; if the length and breadth be increased so as to

number each 4 men more, the whole body will then contain 2320 men. How many men in the militia?

I. Solution by Professor G. B. M. ZERR, Principal of Schools, Staunton, Virginia.



Let $ABCD$ represent the body of militia.

Then $AB: CB=3:2$ $\therefore$ $2AB=3CB$, $AB=\frac{3}{2}CB$

Now $(CB+4)(\frac{3}{2}CB+4)=2320$.

$\therefore$ $CB^2+\frac{20}{3}CB=1536$.

$CB=36$, $AB=\frac{3}{2}CB=54$.

The required number $=36\times54=1944$ men.

II. Solution by P. S. BERG, Apple Creek, Ohio.

1. If 16 men be taken from 2320 men the remainder, 2304 men, may be regarded as standing on 6 squares + 5 rectangles each 4 wide and just as long as each of the six squares.
2. $\therefore$ 1 square + 1 rectangle $\frac{10}{3}$ wide $=2304\div6=384$.
3. $\therefore$ 1 square + 2 rectangles $\frac{5}{3}$ wide $=384$.
4. $\therefore$ Completed square $=384+\frac{25}{9}=386\frac{7}{9}$.
5. $\therefore$ Side of complete square $=19\frac{2}{3}$.
6. $\therefore$ Side of original square $=19\frac{2}{3}-1\frac{2}{3}=18$.
7. $\therefore$ Side rank $=18\times2=36$ men.
8. $\therefore$ Front rank $=18\times3=54$ men.
9. $\therefore$ Total number of men $=54\times36=1944$.

Solved in a similar manner by *J. A. Calderhead*, *H. C. Whitaker*, and *I. L. Beverage*. *A. L. Foote* gave an excellent solution by Algebra.

PROBLEMS.

18. Proposed by L. B. HAYWARD, Superintendent of Schools, Bingham, Ohio.

In a circle whose radius is 6, find the area of the part between parallel chords whose lengths are 8 and 10, both being on the same side of the center.

17. Proposed by H. W. HOLYCROSS, Superintendent of Schools, Pottersburg, Union County, Ohio.

A gentleman owns a circular farm, and if three circles of equal area and as large as possible be drawn within it, the circular area in the center of the farm will contain one acre; what is the area of the circular farm?

16. Proposed by EARL D. WEST, West Middleburg, Logan County, Ohio.

How many stakes can be driven down upon a space 15 feet square allowing no two to be nearer each other than $1\frac{1}{2}$ feet, and how many allowing no two to be nearer than $1\frac{1}{4}$ feet?

15. Proposed by O. S. KIBLER, Superintendent of Schools, West Middleburg, Logan County, Ohio.

Supposing the town A to be 30 mi. from B, B 25 mi. from C, C 20 mi. from A, where must a house be erected to be equally distant from each of the towns?

14. Proposed by Professor G. B. M. ZERR, A. M., Principal of Schools, Staunton, Virginia.

A bank by discounting a note of 7% receives for its money a discount equivalent to $7\frac{3}{4}\%$ interest. How long must the note have been discounted before it was due?

Solutions to these problems should be received on or before May 1st.

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

3. Proposed by Professor H. A. WOOD, A. M., Hoboken, New Jrsey.

If $x^6-y^6=665$, and $x^3y+xy^3=78$, find x and y.

Solution by Professor G. B. M. ZERR, A. M., Principal of Schools, Staunton, Virginia.

Let $x^2-y^2=s$, $xy=p$. Then $x^6-y^6=s^3+3p^2s=665$, $\therefore p^2=\dfrac{665-s^3}{2s}$ ....(1). Also, $p\sqrt{s^2+4p^2}=78$, or $p^2s^2+4p^4=6084$....(2).

(1) in (2) gives, $s^6-3325s^3-54756s^2+1768900=0$.

$\therefore (s-5)(s^5+5s^4+25s^3-3200s^3-70756s^3-353780)=0$.

$\therefore s=5$. $\therefore x^2-y^2=5$, $xy=6$, and now $x=\pm3$, $y=\pm2$.

Solved also by *P. S. Berg*, and Professors *Scheffer* and *Whitaker*.

4. Proposed by L. E. PRATT, Tecumseh, Nebraska.

If Σm, Σm^3, Σm^5, ... Σm^{2n-1} are the sums of the 1st, 3rd, 5th, ... $(2n=1)$th powers of the first m natural numbers, prove that $n\Sigma m^{2n-1}+\dfrac{n(n-1)(n-2)}{\underline{|3}}\Sigma m^{2n-3}+\dfrac{n(n-1)(n-2)(n-3)(n-4)}{\underline{|5}}\Sigma_m^{2n-5}+......=2^{n-1}\Sigma_m^n$.

Solution by the Proposer.

Assume the natural series 1, 2, 3, 4, 5, ...$(m-1)^n$, ...m^n, ...$(m+1)^n$ ....(1). That portion of (1) included between $(m-1)^n$ and $(m+1)^n$ may be written

$\dfrac{\{(m^n-1)-(m-1)\}\{(m-1)^n+m^n\}}{2}$, m_n, $\dfrac{\{(m+1)^n-(m^n+1)\}\{m^n+(m+1)^n\}}{2}$....(2). Collecting all the terms having the form m^n contained in (2), we have $\dfrac{m^n\{(m+1)^n-(m-1)^n\}}{2}$....(3).

This expression is the general term of a series, involving the first m terms of (1), whose sum, as may be seen by changing m into $m-1$, $m-2$, ..., is $\dfrac{m^n(m+1)^n}{2}$, or $2^{n-1}\Sigma_m^n$. If we now expand (3), we have

$$n\,m^{2n-1}+\frac{n(n-1)(n-2)}{\underline{|3}}m^{2n-3}+\frac{n(n-1)(n-2)(n-3)(n-4)}{\underline{|5}}m^{2n-5}+....(4).$$

Each term of (4) may be regarded as a type-term of a series, where m may have the same values as in (3).

$$\therefore\ n\,\Sigma m^{2n-1}+\frac{n(n-1)(n-2)}{\underline{|3}}\Sigma m^{2n-3}+....=2^{n-1}\Sigma_m^n.$$

[By Σ_m^n is meant $(\Sigma_m)^n$. The latter is the preferable way of writing it. Several contributors misunderstood Σ_m^n as given by the proposer.—Editor.]

5. Proposed by WILLIAM E. MAY, 500 Union St., Knoxville, Tennessee.

A, *B*, *C* went to market, each with 10,30, and 50 eggs, respectively. On their

way to market, they agreed to sell their eggs at the same price(s) per dozen so as to realize an equal integral number of cents. How much did they receive?

I. **Solution by A. L. FOOTE, No. 80, Broad St., New York City.**

Let A sell 1 egg at 2 cents, and the remaining 9 at x cents, realizing $(2+9x)$ cents; let B sell $(1+n)$ eggs at 2 cents and $30-(1+n)$, or $29-n$ at x cents, realizing from the two sales $(2+2n+29x-nx)$. But by the conditions we have, $(2+9x)=(2+2n+29x-nx)$, or by solving, $x=\dfrac{2n}{n-20}$; the least value of n to render n integral is 21, consequently $x=42$, and we have, the sales of A and B as follows:

A 1 at 2cts. and 9 at 42cts. =380cts., B 22 at 2cts. and 8 at 42cts. =380cts. Now let C sell m eggs at 2cts. and $(50-m)$ at 42cts., and we have $2m+2100-42m=380$, or $40m=2100-380$, from which $m=43$ eggs at 2cts. and 7 at 42cts., making 380cts., as before. So the sales of each were,

A 1 egg at 2cts. and 9 at 42cts. =380cts.,
B 22 eggs at 2cts. and 8 at 42cts. =380cts.,
C 43 eggs at 2cts. and 7 at 42cts-=380cts.

N. B.—A variety of answers may be obtained by varying the suppositions.

[This egg puzzle problem is somewhat indefinite as published. The proposer afterwards changed the second part of the problem to read: "they agree to sell their eggs at the same rates per dozen, so as to obtain no fractional part of a cent from a sale and receive an equal integral number of cents." He assumes that at first they find the market slow for eggs, and were offered only $\frac{1}{7}$ of a cent per egg, or $1\frac{5}{7}$cts. per dozen. At this price A sells 7, B 28, and C 49. Eggs then take a rise and are worth 3cts. per egg, or 36cts. per dozen, at which price they sell the remainder of their eggs. The sales of A, B, and C then stood as follows:

A 7 eggs at $\frac{1}{7}$cts. (1st sale)+3 at 3cts. (2nd sale)=10cts.,
B 28 aggs at $\frac{1}{7}$cts. (1st sale)+2 at 3cts. (2nd sale)=10cts.,
C 49 eggs at $\frac{1}{7}$cts. (1st sale)+1 at 3cts. (2nd sale)=10cts.

Prof. Whitaker remarks that the price may change nine times, but assuming that it changed once gives the condition that twice the number B sold at the first price equals the number A sold + the number C sold both at that price, a condition expressed by $2y=x+z$.—Editor.]

6. **Proposed by L. E. PRATT, Tecumseh, Nebraska.**

A vessel is to be filled with water by two pipes. The first pipe is kept open during m-nth of the time which the second would take to fill the vessel; then the first pipe is closed and the second is opened. If the two pipes had kept open together, the vessel would have been filled t hours sooner, and the first pipe would have brought in p-qth of the quantity of water which the second pipe really brought in. How long would it take each pipe alone to fill the vessel?

Solution by J. F. W. SHEFFER, A. M., Hagerstown, Maryland.

Let x and y designate the times in which the vessel would be filled by the two pipes, respectively, were each open by itself. The time during which the first pipe is kept open is $=\dfrac{m}{n}y$. $\therefore$ $1-\dfrac{my}{nx}$ is the part of the vessel to be

filled by the second, and $\left(1-\frac{my}{nx}\right)y$ the time which it would take it.

$\therefore \frac{my}{n}+\left(1-\frac{my}{nx}\right)y$=time in which the vessel would be filled. Since $\frac{xy}{x+y}$ is the time during which the vessel would be filled were both pipes kept open,

$$\therefore \frac{my}{n}+\frac{(nx-my)y}{nx}=\frac{xy}{x+y}+t\ldots.(1).$$

If both pipes had been kept open during the time $\frac{xy}{x+y}$, $\frac{y}{x+y}$ would have been the part of the vessel filled by the first pipe,

$$\therefore \frac{y}{x+y}=\frac{p}{q}\left(1-\frac{my}{nx}\right)\ldots.(2).$$

Reducing the equation (2), we get $mpy^2+(mp+nq-np)xy-npx^2=0$.

Solving this with reference to y, a mere quadratic, we get y expressed by x. Let us, for the sake of brevity, put $y=rx$. Substituting this in (1), we obtain a simple equation in x, whence,

$$x=\frac{nt(1+r)}{(m+nr-mr^2)r},\quad y=\frac{nt(1+r)}{m+nr-mr^2}.$$

Solved also by Professor *Whitaker* and the *Proposer*.

7. Proposed by O. S. KIBLER, Superintendent of Schools, West Middleburg, Logan County, Ohio.

A's age equals B's age plus the cube root of C's age; B's age equals C's age plus the cube root of A's age plus 14 years; and, C's age equals the cube root of A's age plus the square root of B's age. What is the age of each?

Solution by J. F. W. SCHEFFER, A. M., Hagerstown, Maryland, and H. C. WHITAKER, B. S., M. E., Philadelphia, Pennsylvania.

Denoting by x^3, y^2, z^3, respectively, the ages of A, B, C, we have the equations $x^3=y^2+z\ldots.(1)$, $y^2=z^3+x+14\ldots.(2)$, $z^3=x+y\ldots.(3)$. By adding (2) and (3) we get $x=\frac{1}{2}(y^2-y-14)$, and from (1), $z=\frac{1}{8}(y^2-y-14)^3-y^2$. Substituting these two values in (3), we get an equation of the 18th degrêe, which it would be a piece of folly to solve, since the only rational values of x, y, z, viz.; $x=3$, $y=5$, $z=2$, can with little trouble be obtained from the original equations.

8. Proposed by H. M. CASH, Salesville, Ohio.

The longer side BC of a field in the form of a parallelogram is a (78) rods; the sum of its shorter side AB, and greater diagonal AC is b (114) rods; the distance from B at right angles with AB to a tree standing on AC, is c (32) rods. Find the area of the field, and the distance from the tree to the corners A, C, and D.

Solution by H. C. WHITAKER, B. S., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

Denote the position of the tree by T, the distance AT by x, CT by y, and AB by $b-x-y$.

$$\text{Now } x=\frac{c}{\cos ATB}=-\frac{c}{\cos BTC}=\frac{2c^2y}{a^2-c^2-y^2}.$$

$$\text{Also } x^2=c^2+(b-x-y)^2,\text{ whence } x=\frac{b^2+c^2-y^2-2by}{2b-2y}$$

Equating these values of x gives,

$y^4-2by^3+(b^2-a^2-2c^2)y^2+2b(a^2+c^2)-(a^2-c^2)(b^2+c^2)=0$. Substituting the

given values of a, b, and c, $y^4-228y^3+4864y^2+1620624y-70941200=0$.

Whence $y=50$ and $x=40$.

$DT=\sqrt{(24^2+50^2-2\times50\times24\times.6)}=\sqrt{1636}=40.4474$ rds.

Area$=2\sqrt{96\times18\times72\times6}=2\times4\times6\times2\times18=1728$ sq. rds.

Solved by Professor *Scheffer*. An excellent trigonometrical solution was also received from *W. L. Harvey*. Portland, Maine.

PROBLEMS.

14. Proposed by SYLVESTER ROBINS, North Branch Depot, New Jersey.

In copying the following example, the class lost the coefficient of x in the equation:

$$\sqrt{x}+\sqrt[3]{y}=x\ldots.(1).$$

$$\sqrt{x^3}+\sqrt[3]{x^2}=(\ \ )x\ldots.(2),$$

and then set themselves to finding coefficients for the vacancy, which would allow rational values to x and y.

15. Proposed by SETH PRATT, C. E., Assyria, Michigan.

From a point in an equilateral triangle, the distances to the angles are, respectively, 20, 28, and 31 rods. Required a side of the triangle.

16. Proposed by COLMAN BANCROFT, Professor of Mathematics, Hiram College, Hiram, Ohio.

A traveller whose speed constantly increases in a geometrical progression passes A at 2 o'clock, B at 3:30, C at 4:30, and D at 6:18. At B he is moving at the rate of 12 miles per hour, and at C 18 miles. Find his rate at A and D, and the distance from A to each of the points B, C, and D.

17. Proposed by G. B. M. ZERR, A. M., Principal of Schools, Staunton, Virginia.

A sum of P dollars is loaned at r per cent. interest. At the end of the first year a payment of x dollars is made; and at the end of each following year the payment is made greater by m per cent. than the preceding year. If the sum is paid in n payments, find x.

18. Proposed by WILLIAM E. HEAL, Member of the London Mathematical Society, Marion, Indiana.

Two railroad trains, lengths m and n, meet at a siding, length l. How shall the trains pass if $l<m<n$?

Solutions to these problems should be received on or before May 1st.

GEOMETRY.

Conducted by B. F. FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

2. Show that $\frac{1}{2}\pi=\left\{\frac{2.\ 4.\ 6.\ 8.\ 10}{1.\ 3.\ 5.\ 7.\ 9}\ldots.\right\}^{\frac{2}{\infty}}$, *Wallis's* expression for π.

II. Solution by P. H. PHILBRICK, C. E., Lake Charles, Louisiana.

The above expression as we will see is not correct.

We have from Trigonometry $\sin x = x\left(1-\frac{x^2}{1.2.3}+\frac{x^4}{1.2.3.4.5}-\text{etc.}\right)$.

To resolve the terms in the parenthesis into factors, we observe that they must be of the form $1-\frac{x^2}{q}$, since the coefficint of x° is 1, and the exponents differ by 2.

Moreover, they must be such as to reduce the second member to zero for $x=\pm n\pi$, or $x^2=n^2\pi^2$, for such values of x reduce the first member to zero. Hence, $q=n^2\pi^2$. Making n successively 1, 2, 3, etc. the above becomes:

$$\sin x = x\left(1-\frac{x^2}{1^2\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\ldots\ldots$$

Putting $x=\frac{\pi}{2}$ we have,

$$\sin\frac{\pi}{2}=1=\frac{\pi}{2}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right)\ldots\ldots$$

$$=\frac{\pi}{2}\left(\frac{2^2-1}{2^2}\right)\left(\frac{4^2-1}{4^2}\right)\left(\frac{6^2-1}{6^2}\right)\ldots\ldots$$

$$=\frac{\pi}{2}\cdot\frac{2-1}{2}\cdot\frac{2+1}{2}\cdot\frac{4-1}{4}\cdot\frac{4+1}{4}\cdot\frac{6-1}{6}\cdot\frac{6+1}{6}\ldots\ldots$$

$$\text{Therefore }\frac{\pi}{2}=\left(\frac{2}{1}\cdot\frac{2}{3}\right)\left(\frac{4}{3}\cdot\frac{4}{5}\right)\left(\frac{6}{5}\cdot\frac{6}{7}\right)\ldots\ldots$$

$$=\left[\frac{2.\ 4.\ 6.\ 8.\ldots.2n}{1.\ 3.\ 5.\ 7.\ldots.2n-1}\right]^2\frac{1}{2n+1}\text{, or}$$

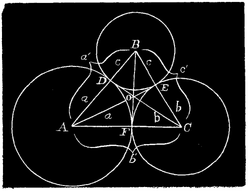
$$=\left[\frac{2.\ 4.\ 6.\ 8.\ldots.(2n)}{1.\ 3.\ 5.\ 7.\ldots.(2n+1)}\right]^2\frac{2n+1}{1}.$$

4. Three circles whose radii are a, b, and c touch each other externally; prove that the tangents at the points of contact meet in a point whose distance from any one of them is $\left\{\frac{abc}{a+b+c}\right\}^{\frac{1}{2}}$. [Selected from *Todhunter's Plane Trigonometry.*]

I. Solution by Professor G. B. M. ZERR, A. M., Principal of Schools, Staunton, Virginia.

Let $OD=OE=OF=r$. Then $\frac{r}{2}(a'+b'+c')=r(a+b+c)=\triangle$ where $\triangle$ = area of triangle; but $\triangle=\sqrt{\frac{1}{2}(a'+b'+c')\frac{1}{2}(a'+b'-c')\frac{1}{2}(a'+c'-b')\frac{1}{2}(b'+c'-a')}=\sqrt{(a+b+c)abc}$. $\therefore\ r(a+b+c)=\sqrt{(a+b+c)\,abc}$.

$$\therefore\ r=\left(\frac{abc}{a+b+c}\right)^{\frac{1}{2}}.$$



This problem was solved in a very excellent manner by *P. S. Berg*, *H. C. Whitaker*, and *P. H. Philbrick*.

II. Solution by Professor J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

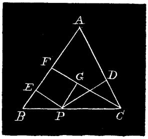
Denote the circles whose radii are a, b, and c respectively by O, O', O'', and P the point of intersection of the tangents common to O, O' and to O', O''. Since P is in the tangent common to O and O', the powers of P with reference to these two circles are equal, and as much as P is in the common tangent of O' and O'', its powers to these two circles are equal; consequently, its powers with ref. to O and O'' being equal, P must lie in the tangent common to O and O''. It is, in fact, the radical center of the three radical axes. Denoting the required distance by x, we have for the areas of the three triangles $OO'P$, $OO''P$, $O'O''P$, respectively, $\frac{1}{2}x(a+b)$, $\frac{1}{2}x(a+c)$, $\frac{1}{2}x(b+c)$; and since the area of $\triangle$ $OO'O''$, whose sides are $a+b$, $a+c$, $b+c$, is $=\sqrt{abc(a+b+c)}$, we have $\frac{1}{2}x(a+b)$ $+\frac{1}{2}x(a+c)+\frac{1}{2}x(b+c)=\sqrt{abc(a+b+c)}$, whence $x=\dfrac{\sqrt{abc(a+b+c)}}{a+b+c}=\sqrt{\dfrac{abc}{a+b+c}}$.

Q. E. D.

5. Proposed by ADOLPH BAILOFF, Durand, Wisconsin.

If from a variable point in the base of an isosceles triangle, perpendiculars are drawn to the sides, the sum of the perpendiculars is constant and equal to the perpendicular let fall from either extremity of the base to the opposite side.

1. Solution by ROBERT J. ALEY, A. M., Professor of Mathematics in the Indiana State University, Bloomington, Indiana.



PG is drawn paralell to AB.

$PE=FG$.

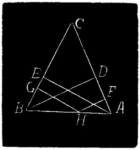
$PD=GC$, because $\triangle$ GPC and $\triangle$ PDC have the same hypotenuse PC and the acute angle GPC of the one equal to the acute angle DCP of the other. $\therefore$ $PE+PD=FG+GC=FC$.

II. Solution by Hon. JOSIAH H. DRUMMOND, LL. D., Portland, Maine;

Connecting the vertex of the triangle with the variable point, wherever in the base it may be, the triangle is divided into two triangles whose area is one-half the product of the perpendicular and one of the equal sides of the triangle respectively; hence, the area of the original triangle is equal to one-half the product of one the equal sides by the sum of the two perpendiculars. Hence the sum of the perpendiculars equals the perpendicular from the extremity of the base to the opposite side.

Solved similarly by *P. H. Philbrick*

III. Solution by A. L. FOOTE, C. E., No. 80, Broad St., New York, and ALBERT J. ROBINSON, Pottersburg, Ohio.



The triangle being isosceles, the angles CAB CBA and are equal, consequently BD and AE are equal, and because they are at right angles to the sides AC and BC, then HF and HG which are at right angles to AC and BC are parallel, HF to BD and HG to AE; whence, $BH: AB:: GH: AE$ and $AH: AB:: FH: BD$. Therefore, $AH+BH: 2AB:: GH+FH: AE+BD$. But $BD=AE$ and $AH+BH=AB$, then we have $AB: 2AB:: GH+FH: 2AE$ or

$GH+FH=\frac{2AE\times AB}{2AB}$ or $=AE$. The same result would follow wherever, in the base AB, the point H is taken so that $FH+HG$ is constant and equal to AE or BD.

Q. E. D.

This problem was also solved in an elegant manner by *J. A. Calderhead*, *Charles E. Myers*, *Professor G. B. M. Zerr*, *P. S. Berg*, and *Professor H. C. Whitaker*.

PROBLEMS.

21. Proposed by CHARLES E. MYERS, Canton, Ohio.

A cistern 6 feet in diameter contains 3 inches of water. If a cylinder, four feet long and one foot in diameter, be laid in a horizontal position on the bottom, to what height will the water rise?

22. Proposed by J. A. TIMMONS, St. Marys, Kentucky.

Given, the perimeter of a triangle $=100(2s)$, the radius of the inscribed circle $=9(r)$, and the radius of the circumscribed circle $=20(R)$; it is required to find (1) the sides of the triangle, (2) the radius of the circle circumscribing the triangle formed by bisecting the exterior angles of the original triangle, (3) the area of the triangle thus formed; all in terms of R, r, s.

23. Proposed by E. L. PRATT, Tecumseh, Nebraska.

The ordinate of the point P of an ellipse is produced to meet the circle described on the major axis as diameter at Q. CQ, the straight line joining Q and the center of the ellipse, is tangent to the circle described on the focal radius of P as diameter.

If θ is the excentric angle of P prove that

$$\sin 2\theta=\frac{2(2a+b)\pm4\sqrt{a(a+b)}}{a-b}$$

24. Proposed by T. W. PALMER, Professor of Mathematics in the University of Alabama.

Two right triangles have the same base, the hypotenuse of the first is equal to 60, of the second 40. The point of intersection of the two hypotenuses is at the distance 15 from the base. Find the length of the base.

25. Proposed by L. B. FRAKER, Weston, Ohio.

The sides of a quadrilateral board are $AB=7$ inches, $BC=15$ inches, $CD=21$ inches, and $DA=13$ inches; radius of inscribed circle is 6 inches. (1) What are dimensions of the largest rectangular board that can be cut out of the given board, (2) largest square, (3) largest equilateral triangle? (Please solve without use of the calculus.)

26. Proposed by J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

$ABCD$ represents a rectangle, and $ABEF$ a trapezoid which is perpendicular to the rectangle, both figures having the side AB common to each other, and ADF and BCE forming two right triangles perpendicular to the rectangle $ABCD$. To determine the conoidal surface $CDFE$ so as to satisfy the condition that any plane laid through AB will intersect it in a straight line. Also find volume of the solid thus formed.

27. Proposed by ADOLPH BAILOFF, Durand, Wisconsin.

A line BE, that bisects an angle exterior to the vertical angle of an isosceles triangle is parallel to the base AC.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

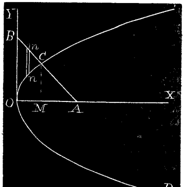
SOLUTION TO PROBLEMS.

I. Find the moment of inertia about the origin, of the area included within the parabola $y^2=4ax$, the line $x+y=3a$, and the axis of x.

[Selected from *Osborne's Differential and Integral Calculus.*]

I. Solution by J. M. BANDY, Professor of Mathematics, Elon College, North Carolina.

The equation of the parabola is $y^2=4ax$, and that of the given line $y=-x+3a$. Combining these equations, we get $x=OM=a$, and $y=CM=2\sqrt{ax}$. $\therefore$ for figure OMC, moment $=\int_0^a\int_0^{2\sqrt{a}}(x^2+y^2)dxdy$.



Again , for fig. MCA, x varies between a and $3a$, and y varies between 0 and $3a-x$.

$$\therefore \text{ Moment}=\int_0^{3a}\int_0^{3a-x}(x^2+y^2)dxdy,$$

$\therefore$ for fig. OCA, moment $=\int_0^a\int_0^{2\sqrt{(ax)}}(x^2+y^2)dxdy+\int_a^{3a}\int_o^{3a-x}(x^2+y^2)dxdy.$

$$=\int_o^{2a}\int_{\frac{y^2}{4a}}^{3a-y}(x^2+y^2)dydx=\frac{314a^4}{35}.$$

II. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics in the University of Mississippi, University, Mississippi.

We wish to find the moment of inertia of AC_nO about OY.

The co-ordinates of C, the point of intersection of $y^2=4ax$ and $x+y=4a$, are $[2a(3-\sqrt{5}),\ 2a(\sqrt{5}-1)]$. Also $OB=OA=4a$. If I, I_1, and I_2 are the moments of inertia about OY of O_nCA, BOA, and BO_nC, respectively, $\mathrm{I}=\mathrm{I}_1-\mathrm{I}_2$. Dividing BO_nC into elementary strips, such as mn, parallel to OY,

$$\mathrm{I}_2=\int_o^{2a(3-\sqrt{5})}(-x+4a-2\sqrt{ax})x^2dx,$$

$$=\tfrac{32}{21}(213-95\sqrt{5})a^4.$$

$$\mathrm{I}_1=\tfrac{1}{6}.\frac{OA^2}{2}.OA^2=\frac{64}{3}a^4.$$

$$\therefore\ \mathrm{I}=\tfrac{64}{3}a^4-\tfrac{32}{21}(213-95\sqrt{5})a^4.$$

$$=\tfrac{32}{21}(95\sqrt{5}-199)a^4.$$

[The equation of the line in *Osborne* is $x+y=3a$, and not $x+y=4a$, as first printed in the MONTHLY. Prof. Bandy solves it for the first case and Prof. Hume for the second. Alsosolved by Professors *H. C. Whitaker*, *P. H. Philbrick*, *J. F. W. Scheffer* and by Cadet *A. J. Vaughan*, of the Virginia Military Institute, Lexington, Virginia.]

2. Proposed by E. S. LOOMIS, A. M., Ph. D., Professor of Mathematics, Baldwin University, Berea, Ohio.

2. Show that the indeterminate form $\dfrac{x-\frac{2}{3}\sin x-\frac{1}{3}\tan x}{x^5}=\dfrac{-1}{20}$, when $x=0$.

[Ex. 51, p. 112, *Williamson's Differential Calculus.*]

I. Solution by C. W. M. BLACK, Department of Mathematics, Wilmington Conference Academy, Dover, Deleware, and G. SERPELL, Student at Virginia Military Institute, Lexington, Virginia.

$$\frac{f'(x)}{\phi'(x)}=\frac{1-\frac{2}{3}\cos x-\frac{1}{3}\sec^2 x}{5x^4}$$

$$\frac{f''(x)}{\phi''(x)}=\frac{\frac{2}{3}\sin x-\frac{2}{3}\sec^2 x\tan x}{20x^3}$$

$$\frac{f'''(x)}{\phi'''(x)}=\frac{\frac{2}{3}\cos x-\frac{4}{3}\sec^2 x\tan^2 x-\frac{2}{3}\sec^4 x}{60\,x^2}$$

$$\frac{f^{\mathrm{IV}}(x)}{\phi^{\mathrm{IV}}(x)}=\frac{-\frac{2}{3}\sin x-\frac{8}{3}\sec^2 x\tan^3 x-\frac{16}{3}\sec^4 x\tan x}{120\,x}$$

$$\frac{f^{\mathrm{V}}(x)}{\phi^{\mathrm{V}}(x)}=\frac{-\frac{2}{3}\cos x-\frac{16}{3}\sec^2 x\tan^4 x-\frac{88}{3}\sec^4 x\tan^2 x-\frac{16}{3}\sec^6 x}{120}$$

$$=\frac{-6}{120}=\frac{-1}{20},\text{ when } x=0.$$

II. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

We have $(x-\frac{2}{3}\sin x-\frac{1}{3}\tan x)\div x^5=x-\frac{2}{3}\left(x-\dfrac{x^3}{6}+\dfrac{x^5}{120}-\text{etc.}\right)$

$-\frac{1}{3}\left(x+\dfrac{x^3}{3}+\dfrac{x^5}{15}+\text{etc.}\right)\div x^5=\dfrac{-1}{20}$, when $x=0$.

Also solved by *Alfred Hume, P. H. Philbrick, J. F. W. Scheffer, H. C. Whitaker G. B. M. Zerr*, and *P. S. Berg*.

3. Proposed by H. C. WHITAKER B. S., M. E., Professor of Mathematics, Manual Training School, Philidelphia, Pennsylvania.

The product of two sides of a triangle is 6,000(k^2), the length of the bisector of the included angle is 60(b). What is the maximum area of the triangle, what is the greatest length of the third side, and what is the total area swept over by the triangle, the bisector remaining fixed? [Selected from *Philadelphia Call*, 26 May, 1890.]

Solution by the Proposer.

Denote one of the including sides by r, the bisected angle by 2θ, the angle the bisector makes with the base by ϕ. The area$=\frac{1}{2}k^2\sin 2\theta$ and is therefore a maximum with $\sin 2\theta$. The segment of the base adjacent to r is $\dfrac{r}{k}\sqrt{k^2-b^2}$ and therefore $k\sin\theta=\sqrt{k^2-b^2}\sin\phi$. From this equation $\sin 2\theta=\dfrac{2b}{k^2}\sqrt{k^2-b^2}$ when $\cos\phi=0$. In this case the triangle is isosceles, the third side$=2\sqrt{k^2-b^2}$ and the area$=b\sqrt{k^2-b^2}$.

By trigonometry, $(k^2-b^2)r^2-2rbk^2\cos\theta+k^2b^2=0$, which is the polar equation of the locus of one end of the base, the bisector being the polar axis and the vertex of the triangle the pole. It is a circle with a center on the

bisector (prolonged) $\frac{bk^2}{k^2-b^2}$ from the vertex and a radius $R=\frac{b^2k}{k^2-b^2}$. The ends of the base move around this circle, the base itself always being a chord passing through a point whose distance from the center is $\frac{b^2k-k^2b+b^3}{k^2-b^2}$. The maximum length of the third side is of course the diameter of this circle $=\frac{2b^2k}{k^2-b^2}$ and the minimum length when the side is at right angles to the bisector$=2\sqrt{k^2-b^2}$.

The area swept through consists of two right triangles with a common hypotenuse and sides k and R $\left(\text{areas}=\frac{b^2k^2}{k^2-b^2}\right)$ and of a sector of circle R with angle at the center$=180°+2\theta\left[\text{Area}=\frac{\pi b^2k(180+2\theta)}{360(k^2+b^2)}\right]$.

The numerical values are as follows:

Maximum Area$=b\sqrt{k^2-b^2}=1200\sqrt{6}=2939.38$.

Greatest length base$=2R=\frac{2b^2k}{k^2-b^2}=163.246$.

Total area swept over$=4899+9022=13921$.

Also solved by *C. W. M. Black, A. H. Bell, J. F. W. Scheffer, G. B. M. Zerr. W. B. Riegner*, of Philadelphia, sent a slip showing an excellent geometrical construction of this problem.

4. Proposed by J. M. COLAW, Principal of High School, Monterey, Virginia.

Three towns A, B, and C are in the same straight line. The distance from A to B is 20 miles and the distance from B to C is 80 miles. A pedestrian started from B for C and traveled at the variable rate of 10 miles an hour reciprocally as the cube root of his distance from A. In what time did he travel from B to C?

Solution by Professor P. H. PHILBRICK, M. S., C. E., Lake Charles, Louisiana, and P. S. BERG, Apple Creek, Ohio.

Let P, distant x from B, be his position at the end of time t.

Then, his velocity at P is $\frac{10}{\sqrt[3]{20+x}}$, and $\frac{dx}{dt}=\frac{10}{\sqrt[3]{20+x}}$,

$$dt=\frac{\sqrt[3]{20+x}}{10}dx\,,\quad t=\tfrac{1}{10}\int_0^{80}\sqrt[3]{20+x}\,dx$$

$$=\tfrac{3}{40}\left[(20+x)^{\frac{4}{3}}\right]_0^{80}=\tfrac{3}{40}\left[(100)^{\frac{4}{3}}-(20)^{\frac{4}{3}}\right]$$

$$=30.74+\text{hours}.$$

Solved by Professors *Hoover, Hume, Scheffer, Whitaker*, and *Finkel*.

5. Proposed by CHARLES E. MYERS, Canton, Ohio.

The volume generated by the curve whose equation is $y^2=px$, revolving about its axis, is cut by a right cylinder whose equation is $y^2=px-x^2$, the axis of the latter passing through the focus of the former. Find the volume common to both by the formula $V=\int\int\int dxdydz$.

I. Solution by Professor G. B. M. ZERR, A. M., Principal of Schools, Staunton, Virginia.

$y^2+z^2=2px$ is the equation of the paraboloid. The limits of z are

$\sqrt{2px-y^2}=z'$ and $-\sqrt{2px-y^2}=z''$. The limits of y are $\sqrt{px-x^2}-y'$ and $-\sqrt{px-x^2}=y''$. The limits of x are o, and p.

$$\therefore\ V=\int_o^p\int_{y''}^{y'}\int_{z''}^{z'}dxdydz=2\int_o^p\int_{y''}^{y'}\sqrt{2px-y^2}\,dxdy,$$

$$=2\int_o^p\left(x\sqrt{p^2-x^2}+2px\sin^{-1}\sqrt{\frac{p-x}{2p}}\right)dx$$

$$=2\left[-\tfrac{1}{3}\sqrt{(p^2-x^2)^3}+px^2\sin^{-1}\sqrt{\frac{p-x}{2p}}-\tfrac{1}{4}px\sqrt{p^2-x^2}+\tfrac{1}{4}\ p^3\sin^{-1}\frac{x}{p}\right]_o^p$$

$$=2p^3\left(\frac{1}{3}+\frac{\pi}{8}\right)=\tfrac{1}{12}p^3(8+3\pi).$$

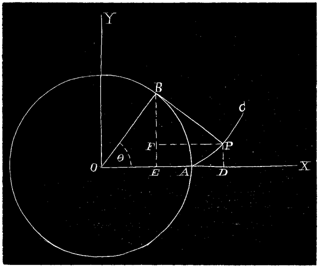
Also solved by *Professor Hume* and the *Proposer*.

6. Proposed by O. S. KIBLER, Superintendeut of Schools, West Middleburg, Ohio.

A string is wound spirally twenty times around a cylinder 20 feet high and 2 feet in diameter. Through what distance will a dove fly in unwinding the string keeping it tense at all times (1) flying in the same plane and (2) not flying in the same plane?

Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University P. O., Mississippi.

Let the circle whose center is O and the curve AC be, respectively, the horizontal projections of the vertical cylinder and the dove's path.



Let P be any point of AC and A the original position of that end of the string to which the dove is attached.

Case I: THE DOVE FLYING IN ONE PLANE.

AC will be the involute of the circle. Denoting the angle AOB by θ and OA by a, $BA=a\theta=BP$.

The co-ordinates of P refered to axes as shown in the figure are,

$x=OD=OE+FP=a\cos\theta+a\theta\sin\theta$, $y=PD=BE-BF=a\sin\theta-a\theta\cos\theta$.

Therefore, $dx=(-a\sin\theta+a\theta\cos\theta+a\sin\theta)d\theta=a\theta\cos\theta d\theta$.

$dy=(a\cos\theta+a\theta\sin\theta-a\cos\theta)d\theta=a\theta\sin\theta\, d\theta$.

$ds^2=dx^2+dy^2=a^2\theta^2d\theta^2$; $ds=a\theta d\theta$; $s=\int a\theta d\theta=a\frac{\theta^2}{2}$, s being estimated from A.

$a=1$, and the θ-limits are 40π and 0. $\therefore$ $s=800\pi^2=7895$ feet, approximately.

Case II: THE STRING BEING KEPT HORIZONTAL.

BP, now, is the length of that part of the string unwound. Hence, BP is the hypotenuse of aright triangle whose base is the arc AB and whose altitude is

the vertical distance of the dove from A. This may be represented by $c\theta$. It follows that $BP=\sqrt{a^2\theta^2+c^2\theta^2}=(a^2+c^2)^{\frac{1}{2}}\theta$.

The co-ordinates of P are:

$x=a\cos\theta+(a^2+c^2)^{\frac{1}{2}}\theta\sin\theta$,

$y=a\sin\theta-(a^2+c^2)^{\frac{1}{2}}\theta\cos\theta$,

$z=c\theta$, the axis of z coinciding with the cylinder's axis. $\therefore\ ds^2=dx^2+dy^2+dz^2=\left[2a^2+2c^2-2a\ (a^2+c^2)^{\frac{1}{2}}+(a^2+c^2)\theta^2\right]d\theta^2$.

$$\therefore\ s=\int\left[2a^2+2c^2-2a(a^2+c^2)^{\frac{1}{2}}+(a^2+c^2)\theta^2\right]^{\frac{1}{2}}d\theta=\sqrt{a^2+c^2}\left\{\frac{\theta}{2}\left(\frac{2a^2+2c^2-2a(a^2+c^2)^{\frac{1}{2}}}{a^2+c^2}+\theta^2\right)^{\frac{1}{2}}+\frac{a^2+c^2-a\sqrt{a^2+c^2}}{a^2+c^2}\log\left[\theta+\left(\frac{2a^2+2c^2-2a\sqrt{a^2+c^2}}{a^2+c^2}+\theta^2\right)^{\frac{1}{2}}\right]\right\}.$$

$a=1$, $c=\dfrac{1}{2\pi}$, and the θ-limits are 40π and 0.

$\therefore\ s=7995$ feet, approximately.

[*C. E. Myers* solves for case I. and gets 7895.517768 feet. *Prof. Zerr* solves by a different method and gets 8023.754 feet and 8124.910+feet, for his results.]

PROBLEMS.

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12. Proposed by ISAAC L. BEVERAGE, Monterey, Virginia.

Given the equations $2z^3=x+3z$ and $5z^2=y+2z$. To find $\dfrac{dy}{dx}$ for $x=o$.

13. Proposed by J. A. CALDERHEAD, B. Sc., Superintendent of Schools, Limaville, Ohio.

A steamer whose course is due west and speed 10 knots is sighted by another steamer going at 8 knots; what course must the latter steer, so as to cross the track of the former at the least possible distance from her?

14. Proposed by J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

Right triangles are inscribed in a circle whose centre $=(a,b)$, and radius$=c$. If one of the legs passes through a fixed point, prove that $c^2(x^2+y^2)=(a^2+b^2-c^2-ax-by)^2$ is the curve to which the other leg is always tangent; the fixed point being the origin of the co-ordinates.

15, Proposed by CHARLES E. MYERS, Canton, Ohio.

From a given quantity of material a cylindrical cup with circular bottom and open top is to be made, the cup to contain the greatest amount. What must be its dimensions?

Solutions to these problems should be received on or before May 1st.

MECHANICS.

Conducted by B. F. FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

1. Proposed by A. M. SCRIPTURE, A. M., Principal of Schools, New Hartford, New York.

Suppose A and B to start at the same time from the same point in a level plane. A walks south, going 30 inches at a step, stepping twice in a second and beating a drum at the beginning of each step. B steps west 30 inches every time he hears the drum beat. How far apart will they be at the end of 15 minutes, if the temperature of the air is $+41°$ Fahrenheit?

Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

The man who hears the drum beat always hears it at the extremity of the hypotenuse of an isosceles right triangle.

The length of the first hypotenuse is $\frac{30\sqrt{2}}{12}=\frac{5\sqrt{2}}{2}$ feet, of the second $\frac{60\sqrt{2}}{12}=\frac{10\sqrt{2}}{2}$ feet, of the third $\frac{15\sqrt{2}}{2}$ feet, and so on. Let a be the number of feet sound travels per second. Then $\frac{1}{2}+\frac{5\sqrt{2}}{2a}$ = the number of seconds from first beat till he hears the second beat;

$\frac{2}{2}+\frac{10\sqrt{2}}{2a}$ from the first till he hears the third beat;

$\frac{3}{2}+\frac{15\sqrt{2}}{2a}$ from the first till he hears the fourth beat;

$n\left(\frac{1}{2}+\frac{5\sqrt{2}}{2a}\right)$ from the first till he hears the $(n+1)$th beat.

But $\frac{n}{60}\left(\frac{1}{2}+\frac{5\sqrt{2}}{2a}\right)=15$ or $n=\frac{1800a}{a+5\sqrt{2}}$. $41°F=\frac{5}{9}(41-32)=5°C$. Velocity of sound at $0C°$ and 760 mm. pressure $=1090$, hence velocity at $5C$ and 760 mm. pressure $=1090\sqrt{\frac{273+5}{273}}=1099.81$ feet.

$\therefore a=1099.81$ feet. $\therefore n=1780.5$.

$\therefore$ he takes 1781 steps plus the first step $=1782$.

$1782\times2\frac{1}{2}$ feet $=4455$ feet $=B$'s distance from starting point.

$1800\times2\frac{1}{2}$ feet $=4500$ feet $=A$'s distance from starting point.

d = distance apart $=\sqrt{(4455)^2+(4500)^2}=6332.22$ feet.

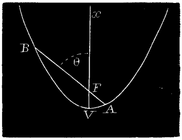
[*Professor H. C. Whitaker* gets 6355.06 feet.]

2. Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy, Pacific College, Santa Rosa, California.

The axis of a parabola coincides with a vertical line. What is the position of that focal chord through which a body would roll down in the least time?

I. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University P. O., Mississippi.

Let AB be a focal chord of the parabola AVB, making an angle θ with Vx, the vertical axis. If $2p$ is the latus rectum of the parabola, its polar equation is $r=\dfrac{p}{1-\cos\theta}$.



Then $FB=\dfrac{p}{1-\cos\theta}$, $FA=\dfrac{p}{1+\cos\theta}$,

$AB=FB+FA=\dfrac{2p}{1-\cos^2\theta}=\dfrac{2p}{\sin^2\theta}$.

If t be the time of descent and g the acceleration due to gravity,

$\dfrac{2p}{\sin^2\theta}=\frac{1}{2}g\cos\theta t^2$, and $t=\sqrt{\dfrac{4p}{g}}\cdot\dfrac{1}{\sin\theta\sqrt{\cos\theta}}$.

Equating to zero the first derivative of t with regard to θ and solving the resulting equation, $2\cos^2\theta-\sin^2\theta=0$, $\sin\theta=\sqrt{\frac{2}{3}}$, or $\theta=\tan^{-1}\sqrt{2}$, and this, evidently, is the value of θ for the *minimum* value of t.

II. Solution by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy, Pacific College, Santa Rosa, California.

Take 1st the two central curves with the transverse axes vertical. Let θ be the angle required. From the polar equations,

$$\text{the focal chord } l=\begin{cases}\dfrac{2a(1-e^2)}{1-e^2\cos^2\theta} & \text{in the ellipse,}\\[2mm] \dfrac{2a(e^2-1}{1-e^2\cos^2\theta} & \text{in the hyperbola.}\end{cases}$$

If t be the time required for the body to roll through l,

$$\text{then } t^2=\frac{2l}{g\cos\theta}=\begin{cases}\dfrac{4a}{g}\cdot\dfrac{1-e^2}{\cos\theta-e^2\cos^3\theta} & \text{in the ellipse,}\\[2mm] \dfrac{4a}{g}\cdot\dfrac{e^2-1}{\cos\theta-e^2\cos^3\theta} & \text{in the hyperbola.}\end{cases}$$

$$\therefore\ t\frac{dt}{d\theta}=\begin{cases}\dfrac{2a(1-e^2)}{g}\cdot\dfrac{\sin\theta-3e^2\cos^2\theta\sin\theta}{(\cos\theta-e^2\cos^3\theta)^2} & \text{in the ellipse,}\\[2mm] \dfrac{2a(e^2-1)}{g}\cdot\dfrac{\sin\theta-3e^2\cos^2\theta\sin\theta}{(\cos\theta-e^2\cos^3\theta)^2} & \text{in the hyperbola.}\end{cases}$$

In order that t should be a minimum, $\sin\theta-3e^2\cos^2\theta\sin\theta=0$ in both curves. $\therefore\cos^2\theta=\dfrac{1}{3e^2}$, $\sin^2\theta=\dfrac{3e^2-1}{3e^2}$. Making $e=1$,

in the parabola, $\theta=\sin^{-1}\sqrt{\frac{2}{3}}=54^\circ\ 44'\ 8''$ required.

[NOTE.— (1) Making $e>1$, any value assigned will give possible values for θ in the hyperbola. If e becomes indefinitely great θ approaches a right angle. (2) Making $e<1$ for all values of $e>\sqrt{\frac{1}{3}}$ and <1, θ is some possible angle. When $e=\sqrt{\frac{1}{3}}$, the focal chord required, coincides with the transverse axis. For all values when $e<\sqrt{\frac{1}{3}}$ $\sin\theta$ is imaginary, $\cos\theta$ takes impossible values.

Hence, in this case, θ has no value and the transverse axis must be taken as the only focal chord satisfying the conditions required.]

PROBLEMS.

9. Proposed by CHARLES E. MYERS, Canton, Ohio.

A ladder inclined at an angle of 60° to the horizon rests with one end on a *rough* pavement and the other against a *smooth* vertical wall. If the coefficient of friction between the foot of the ladder and the pavement is $\frac{1}{6}\sqrt{3}$, to what height can a man ascend before the ladder will begin to slip?

10. Proposed by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

A paraboloid floats in a liquid which fills a fixed paraboloidal shell: both the paraboloid and the shell have their axes vertical and their vertices downward: the latus rectum of the paraboloid and shell are equal, and the axis of the shell is m times that of the paraboloid. If the paraboloid be pressed down until its vertex reaches the vertex of the shell, so that some of the liquid overflows. and then released, it is found that the paraboloid rises until it is just wholly out of the liquid, and then begins to fall. Prove that (1) the densities of the paraboloid and liquid are in the ratio

$2[m^2+m+1-(m+1)\sqrt{(m^2-1)}]$: $3\sqrt{(m+1)\div(m-1)}$, the free surface of the liquid being supposed to remain horizontal throughout the motion; and (2) if cone and conical be used, the ratio is $3[m^4-1-(m^3-1)\sqrt[3]{(m^3-1)}]:4\sqrt[3]{(m^3-1)}$, the vertical angles being equal.

Solutions to these problems should be received on or before May 1st.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTION TO THE CELEBRATED INDETERMINATE EQUATION.

$$x^2-Ny^2=\pm1.$$

By A. H. BELL Hillsboro, Illinois.

[Continued from February Number.]

It will already be noted, that whenever the denominator of y in (A) or the converging difference (as it is called) becomes $+1$, then the denominator $m^2N-n^2=+1$, by changing signs etc. is $n^2-Nm^2=-1$, and when the converging difference is -1, gives $n^2-Nm^2=+1$. The n and m in both cases occupy the position of x and y, in the original equation. Formula (A) giving the second higher values as x_2 and y_2 these answering always the $+1$ condition, and we now have the germ, of the formula as given by Legendre, to obtain the 2nd values after the 1st are obtained. This seems to be the extent to which the higher values of x and y were carried, except by actual computations.

$x^2-Ny^2=-1$, this class of numbers N, can always be known by this

converging difference, being $+1$. They are also known when the number of fractions in the cycle, are odd, from $\sqrt{N}$.

Again the majority of them can be recognized, when of the form of a^2+1, or $4m+1$, without regard to prime numbers especially.

Formula (A), in the value given for y, the denominator m^2N-n^2 must be a *factor* of $2\times m\times N$ and the factor required from N must also be one of n. When m is made successively 1, 2, 3,etc. and substituted in m^2N-n^2,the resulting nearest values of n are called the $+$ or $-$ *Convergent Difference*. The small differences appear in cycles, and they can be developed in a series of two differences, while the corresponding m and n values proceed in an arithmetical series. Then we have a triple series; and a term is m, n, and convergent difference. In making m=to 1, 2, 3, etc. and a factor of $2\times m\times N$ etc. not being found, the first comparatively small $+$ or $-$ convergent difference of the advance terms is selected and multiplied by 2 becomes D_2, of the second order of differences, for the convergent diff. series. The m and n belonging to the selected convergent difference constitute the *common difference* for the two series of m's and n's, up to this time for every value given to m and the resulting two n's with their two differences give two terms with the same m.

Two terms of the Triple Series are now selected from these sets of preceding terms, so that when the m's and n's are added together will equal m and n already chosen for the common differences. In commencing a series, the two terms will always have contrary signs, the term having the same sign of the D_2 selected will constitute the 1st term of the Triple Series. With what is now given the convergent difference series is developed until it changes signs, and a limit is reached. The number of terms now being found,the m and n series is then carried forward, and if at the limit reached the convergent differences fail to become a factor required; then a New Series is made using the last term for the D_2 etc. provided the two last differences are nearly equal. If not, use the term having the small difference and $2\times$ convergent difference$=D_2$. Proceeding the same as at first and continuing the operation, until the required factor is found, when the values of x and y are quickly found in whole numbers.

No 1. Example. Given $x^2-13y^2=\pm1$, to find x and y in integers.

Value N	m^2	n^2+ Diff.	n^2- Diff.	No. Terms	1	2	3	4	
							+2		
$13=13\times1^2$	$=3^2$	$+\ 4$	$=4^2-3$	$m=$	1	+1	3	5	and $y=180$
							+7		$x=649$
$52=13\times2^2$	$=7^2$	$+\ 3$		$n=$	3	+4	11	18	
		Convergent series	of Diff.		+4	-3	-4	+1	

1st order of differences $=$ .... $D_1=-7-1+5$

The conv. diff. for,$m=2$,and $n=7$,$=+3$ and$\times2=D_2=+6+6+6$

Proof: $421201-13\times180^2=+1$

The conv. diff. being $+1$, x and y are 18 and 5, as shown for the -1 condition.

No. 2. Example: Given $x^2-19y^2=1$, to find x and y.

Value	N	m^2	n^2+Diff.	n^2-Diff.
19	$=19\times$	1^2	$=4^2+3$	$=5^2-6$
76	$=19\times$	2^2	$=8^2+12$	$=9^2-5$
171	$=19\times$	3^2	$=13^2+2$	

This gives at once $\left.\begin{array}{r} m=3 \\ n=13 \\ \text{Diff.}\ \ +2 \end{array}\right\}$ and in(A) we have $y=39$. $x=180$.

Proof: $28900-19\times1521=1$.

No. 3. Example: $x^2-31y^2=1$, to find x and y.

Value N m^2 n^2+ Diff. n^2-Diff.

$31=31\times1=5^2+6^2=6-5$

$124=31\times2^2=11^2+3$

$279=31\times3^2=\dots\dots\dots\dots17^2-10$

$496=31\times4^2=22^2+12$

$775=31\times5^2=\dots\dots\dots\dots28^2-9$

Terms No.	1	2	3
		$+5$	
Take $m=$	$3+$	2	7
		$+28$	
$n=$	$17+$	11	39
Diff.	-10	$+3$	-2
D_1	$+13$	-5	
D_2	-18	$\dots-18$	

give $y=273$, $x=1520$

(TO BE CONTINUED.)

1. Proposed by EARL D. WEST, West Middleburg, Logan County, Ohio.

It is required to divide a given square number into two such parts that each part will be a square number.

I. Solution by Hon. JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

No general solution can be given because there are *comparatively* few square numbers which are severally equal to the sum of two other squares. The well known equation $(2pq)^2+(p^2-q^2)^2=(p^2+q^2)^2$, in which $2pq$, p^2-q^2, and p^2+q^2 represent the sides of right triangles, is believed to include the cases in which the sum of two squares equals a third square. If this is true, it follows that the square root of the given square $(p^2+q^2)^2$ must also be the sum of two squares, p^2+q^2; and we have $p^2+q^2=$the square root of the given number to find $(2pq)^2$ and p^2-q^2, p being $>q$. If this can be done only by trial let us determine the limits within which the trials may be confined. Let m be the square root of the given number, and p and q may be any part positive numbers the sum of whose squares$=m$. Also, if any factor of m is equal to the sum of two squares, say s^2 and t^2, p and q may be s and t multiplied, respectively, by the corresponding factor, but, if any value of p is not prime to the corresponding value of q, the value obtained by some of the processes may be the same as the value obtained by another process. If $p^2+q^2=m$, remembering that $p>q$, we have $p<\sqrt{m}$ and $p<\sqrt{\frac{m}{2}}$. Take $m=82$, then $p<10$ and $p>6$; that is, p may be 7, 8, or 9, but $=7$ or 8 gives impossible values for q; while $p=9$, gives $q=1$. Then $2pq=18$ and $(p^2-q^2)=80$. Hence 18^2 and 80^2 are the numbers, and we have $18^2+80^2=82^2$. But $82=2\times41$; then $s<7$ and $s>4$, and so may be 5

or 6; but $s=6$ makes t impossible, while $s=5$ makes $t=4$, and we have $2st=40$ and $(s^2-t^2)=18$. So $2pq=80$ and $(p^2-q^2)=18$, as before. Take $m=50$, p may be 6 or 7, but 6 makes q impossible, while $p=7$ gives $q=1$, and $2pq=14$ and $p-q^2=48$. $\therefore$ $14^2+48^2=50^2$. But $50=5\times 10$, and $5=4+1$; then $s=2$ and $t=1$, $2st=4$, and $s^2-t^2=3$, $2pq=40$, and $p^2-q^2=30$, and $30^2+40^2=50^2$. Solutions may be obtained from other factors, but they give one of these two results. If $m=65$, it may be shown in the same manner that there are 4 pairs of numbers answering the conditions.

II. Solution by R. J. ADCOCK, Larchland, Illinois.

Since $(x^2+y^2+z^2+u^2+v^2)^2=(x^2+y^2+z^2+u^2+v^2)^2+(2xv)^2+(2yv)^2+(2zv)^2+(2uv)^2$, is true for two or any greater number of letters. Then

$$\left[(2x+1)^2+(2y)^2\right]^2=\left[(2x+1)^2-(2y)^2\right]^2+\left[2(2x+1)2y\right]^2.$$

$$\text{And, } 1=\left[\frac{(2x+1)^2-(2y)^2}{(2x+1)^2+(2y)^2}\right]^2+\left[\frac{2(2x+1)2y}{(2x+1)^2+(2y)^2}\right]^2.$$

If n^2 be the given square number,

$$n^2=\left[\frac{(2x+1)^2-(2y)^2}{(2x+1)^2+(2y)^2}\times n\right]^2+\left[\frac{2(2x+1)2yn}{(2x+1)^2+(2y)^2}\right]^2 \text{ are the}$$

two square parts into which n^2 is divided. Observing that x may have any value including 0, y any value not $=0$, and in order to avoid repetion of parts into which the square number is divided, the numbers for x and y must nót make $2x+1$ and $2y$ have a common factor or be equal. If $x=0$, and $y=1$, the two square parts of n^2 are $\left(\frac{3n}{5}\right)^2$ and $\left(\frac{4n}{5}\right)^2$.

Also solved by *H. W. Draughon*, *G. B. M. Zerr*, *P. S. Berg*, *A. L. Foote*, *P. H Philbrick*, and *H. C. Whitaker*.

2. Proposed by J. M. COLAW, Principal of High School, Monterey, Virginia.

Find two numbers, such that the difference of their squares may be a cube, and the difference of their cubes a square.

I. Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let $6x^3$, $10x^3$ be the numbers.

Then $100x^6-36x^6=64x^6=$ a cube number, and $1000x^9-216x^9=784x^9=(28)^2x^9=$ a square number, when x is a perfect square. Let $x=1$, then 6 and 10 are the numbers. Other values can be obtained by substituting 4, 9, 16, etc. for x.

II. Solution by C. W. M. BLACK, Department of Mathematics in Wilmington Conference Academy, Dover, Delaware.

(A). Represent the numbers by x, and $x+a$. The difference of the cubes will be (1), $a(3x^2+3ax+a^2)$, and the difference of the squares (2), $a(2x+a)$. For one solution the first will be a square if $a=3x^2+3ax+a^2$, and the second a cube if $a=(a+2x)^2$. Solving these equations, $a=1$ and $x=-1$ or 0, $a=0$ and $x=0$, making the required numbers -1 and 0, 0 and 1, or 0 and 0. Also, $a(2x+a)$ will be a cube if (3), $a^2=a+2x$. Combining (1) and (3), $(a-1)(3a^2+3a+4)=0$. The first factor gives $a=1$, whence $x=-1$ or 0, as before. The second factor gives an imaginary result. As these results are not satisfactory, we seek another method for finding other values, if there are any.

(B). It may easily be shown that if x and $x+a$ are positive intergers

and prime to each other, there is in general no solution. So, we have only to find how two numbers not prime to each other may be obtained to fulfill the requirements

(C). Let n and p be two positive integers prime to each other, then (4), $n^3-p^3=hr^2$, and (5), $n^2-p^2=ks^3$, where r, s, h, k, are positive integers. Let m be a factor such that if n and p are both multiplied by it, they will fulfill the required condition. Then, (6), $m^3n^3-m^3p^3=m^3hr^2=$ a square. (7), $m^2n^2-m^2p^2=m^2ks^3=$a cube. The first will be a square if $m=hc^2$ (8), and the second a cube if $m=kc^3$ (9). Combining(8)and (9),$c=\frac{h}{k}$and $m=\frac{h^3}{k^2}$ (10).

Then by taking any two positive integers prime to each other, substituting, and finding values of h and k in equations (4) and (5), and multiplying each member by the value of m from (10),we have two numbers which fulfill the requirements. There is nothing in above proof to limit the method to positive integers prime to each other,but these may be taken as the basis of all positive numbers. When one of the numbers taken is negative,different results are derived. When both are negative, the value derived for m will change the numbers so that the difference of the cubes will not be negative. (Ex. 1) $5^3-3^3=98=2\times7^2$, $5^2-3^2=16=2\times2^3$, $h=2$, $k=2$, $m=2$. $\therefore$ numbers are 10 and 6. (Ex. 2). $2^3-1^3=7$, $2^2-1^2=3$, $h=7$, $k=3$, $m=\frac{7^3}{3^2}$. $\therefore$ numbers are $\frac{686}{9}$ and $\frac{343}{9}$. (Ex. 3). $2^3-(-1)^3=9$, $2^2-(-1)^2=3$, $h=1$, $k=3$, $m=\frac{1}{9}$. $\therefore$ numbers are $\frac{2}{9}$ and $-\frac{1}{9}$. (Ex. 4). $1^3-0^3=1$, $1^2-0^2=1$, $h=1$, $k=1$, $m=1$. $\therefore$ numbers are 1 and 0,as derived before in (A).

Also solved by *H. W. Draughon*, *A. L. Foote*, *P. H. Philbrick*, and *G. B. M. Zerr*.

3. Proposed by O. S. KIBLER, Superintendent of Schools, West Middleburg, Logan County, Ohio.

It is required to find three whole numbers in an arithmetical progression, such that the sum of every two of them shall be a square.

Solution by A. L. FOOTE, 80 Broad St., New York City.

Let $\frac{1}{2}x^2-y$, $\frac{1}{2}x^2$, and $\frac{1}{2}x^2+y$ be the numbers. Then we must have x^2, x^2+y, and x^2-y each squares and since x^2 is a square, we require only to make x^2+y and x^2-y squares. Let $x^2+y=m^2$ and $x^2-y=n^2$. Let $y=2sx+s^2$ and we have $x^2+y=x^2+2sx+s^2=(x+s)^2$, and so we have but to find $x^2-2sx-s^2=n^2$. Now let $x^2-2sx-s^2=(x-m)^2=x^2-2mx+m^2$ and we have $x=\frac{m^2+s^2}{2m-2s}$ where m and s may be assumed at pleasure. Let $m=5$ and $s=4$, then $x=\frac{25+16}{10-8}=\frac{41}{2}$ and $\frac{1}{2}x^2=\frac{1681}{8}$, $y=2sx+s^2=164+16=180$, so we have $\frac{1681}{8}$, $\frac{241}{8}$, and $\frac{3121}{8}$. To render these integral multiply each by 16 and we have 482, 3362 and 6242 for the required numbers. The squares are $3844=62^2$, $6724=82^2$. and $9604=98^2$. The values of m must be so chosen that $\frac{1}{2}x^2-y$ will be positive.

J. H. Drummond finds 380, 8450 and 16514. Also solved by *P. S. Berg* and *H. W. Draughon*.

4. **Proposed by H. W. Holycross, Superintendent of Schools, Pottersburg, Union County, Ohio.**

What value of x will render $4x^4+12x^3-3x^2-2x+1$ a square?

Solution by P. S. BERG, Apple Creek, Ohio.

Extracting the square root of the expression we get $2x^2+3x-3$ as the partial root and a remainder of $16x-8$, or the expression $=(2x^2+3x-3)^2+16x-8$. Now when $16x-8=0$, the expression is a square. $\therefore$ $x=\frac{1}{2}$ is the required value.

Also solved by *J. H. Drummond, A. L. Foote, H. C. Whitaker, G. B. M. Zerr*, and *O. S. Kibler*.

PROBLEMS.

5. **Proposed by ISAAC L. BEVERAGE, Monterey, Virginia.**

Find three numbers the sum of the squares of any two of which diminished by their product shall be a square number.

6. **Proposed by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.**

Find three whole numbers the sum of any two of which is a cube.

8. **Proposed by Hon. JOSIAH H. DRUMMOND, Portland, Maine.**

Every odd square is of the form $4a+1$; find the value of a for the nth consecutive odd square.

Solutions to these problems should be received on or before May 1st.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

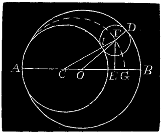
SOLUTION TO A PROBLEM IN AVERAGE.

By B. F. FINKEL, A. M., Professor of Mathematics in Kidder Institute, Kidder, Missouri.

Two circles whose radii are R and r respectively, are tangent internally. Find the average area of all circles that can be drawn tangent to the two circles.

Let $AO=R$ be the radius of the larger circle; $AC=r$, the radius of the smaller circle; and A the point of internal tangency of the two circles.

Let ID be any circle inscribed within the crescent, F the center of this circle, $IF=z$, its radius, and (x, y) the rectangular co-ordinates of the point F referred to A as the origin of co-ordinates. Draw FC and OD. Then $OF=(R-z)$, $CF=(r+z)$, $CE=(x-r)$, and $OE=(x-R)$. From the figure, we have $FC^2-CE^2=OF^2-OE^2$, or $(r+z)^2-(x-r)^2=(R-z)^2-(x-R)^2$, whence $z=\left(\frac{R-r}{R+r}\right)x$. We also have $OF^2=OE^2+EF^2$, or $(R-z)^2=(x-R)^2+y^2$. Substituting the value of z and solving with respect to y^2, we have $y^2=\frac{4Rr}{(R+r)^2}[(R+r)x-x^2]$,



which is the rectangular equation of an ellipse referred to its left hand vertex as origin. Changing this to an equation referred to the center of the ellipse and substituting a for $\frac{1}{2}(R+r)$ and b for $\sqrt{Rr}$, we have $y^2+(1-e^2)x^2=a^2(1-e^2)$. $z=\left(\frac{R-r}{R+r}\right)[\frac{1}{2}(R+r)+x]=e(a+x)$, referred to the center of the ellipse.

The centers of the inscribed circles are uniformly distributed on the circumference of this ellipse, and the number of circles is, therefore, proportional to the circumference of the ellipse.

Let s = the length of any portion of the circumference of the ellipse, measured from the extremity of the transverse diameter.

Then the *average area* sought is

$$\varDelta=\frac{\int \pi z^2 ds}{\int ds}. \quad \text{But } ds=\frac{dx}{(a^2-x^2)^{\frac{1}{2}}}\times(a^2-e^2x^2)^{\frac{1}{2}} \text{ and } z=e(a+x),$$

$$\text{where } e=\left(\frac{R-r}{R+r}\right). \quad \therefore \quad \varDelta=\frac{\pi e^2}{aE(e)}\int_{-a}^{a}\frac{(a+x)^2dx}{\sqrt{(a^2-x^2)}}\times(a^2-e^2x^2)^{\frac{1}{2}}$$

$$=\frac{\pi e^2}{aE(e)}\int_{-a}^{a}\left[\frac{a^2(a^2-e^2x^2)^{\frac{1}{2}}}{\sqrt{(a^2-x^2)}}+\frac{2ax(a^2-e^2x^2)^{\frac{1}{2}}}{\sqrt{(a^2-x^2)}}+\frac{x^2(a^2-e^2x^2)^{\frac{1}{2}}}{\sqrt{(a^2-x^2)}}\right]dx,$$

$$=\frac{\pi e^2}{aE(e)}\left[a^3E(e)+\int_{-a}^{a}\frac{2ax(a^2-e^2x^2)^{\frac{1}{2}}dx}{\sqrt{(a^2-x^2)}}+\int_{-a}^{a}\frac{x^2(a^2-e^2x^2)^{\frac{1}{2}}dx}{\sqrt{(a^2-x^2)}}\right].$$

The value of the second term [in the above equation] is zero between the limits $x=-a$ and $x=a$. If we let $x=av$, we have for the third term,

$$\int_{-a}^{a}\frac{x^2(a^2-e^2x^2)^{\frac{1}{2}}dx}{\sqrt{(a^2-x^2)}}=a^3\int_{-1}^{1}\frac{v^2(1-e^2v^2)^{\frac{1}{2}}dv}{\sqrt{(1-v^2)}}$$

$$=a^3\left[\int_{-2}^{1}\left\{\frac{(1-e^2v^2)^{\frac{1}{2}}}{\sqrt{(1-v^2)}}-(1-v^2)^{\frac{1}{2}}(1-e^2v^2)^{\frac{1}{2}}\right\}dv\right]$$

$$=a^3\left[E(e)-\frac{[1-(1+e^2)v^2+e^2v^4]dv}{\sqrt{(1-v^2)}\sqrt{(1-e^2v^2)}}\right]=a^3\left[E(e)-\tfrac{1}{3}[v(1-v^2)(1-e^2v^2)^{\frac{1}{2}}\right]_{-1}^{1}$$

$$+\tfrac{1}{3}\int_{-1}^{1}\frac{[2-(1+e^2)v^2]dv}{\sqrt{(1-v^2)}\sqrt{(1-e^2v^2)}}\Bigg]=a^3\left[E(e)+\frac{1}{3e^2}(1+e^2)\int_{-1}^{1}\frac{(1-e^2v^2)^{\frac{1}{2}}dv}{\sqrt{(1-v^2)}}\right.$$

$$\left.-\frac{1}{3e^2}\int_{-1}^{1}\frac{dv}{\sqrt{(1-v^2)}\sqrt{(1-e^2v^2)}}\right]=a^3\left[E(e)-\frac{2}{3e^2}(1+e^2)E(e)+\tfrac{2}{3}(1-e^2)F(e)\right].$$

$$\therefore \quad \varDelta=\frac{\pi e^2}{aE(e)}\left\{a^3E(e)+{}^3a\left[E(e)-\frac{2}{3e^2}(1+e^2)E(e)+\tfrac{2}{3}(1-e^2)F(e)\right]\right\},$$

$$=\frac{\pi}{3}a^2\left[4e^2-2+(1-e^2)\left(\frac{F(e)}{E(e)}\right)\right]=\frac{\pi}{3}\left[(R-r)^2-Rr+Rr\left(\frac{F(e)}{E(e)}\right)\right].$$

1. Proposed by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Three persons A, B, C, throw with three dice. They each stake $10.00 and the one who first throws at least ten with the three dice takes the whole stake. Find the expectation of each.

Solution by the Proposer.

The chance of throwing respectively 10, 11, 12, ... ,18, with three dice is $\frac{27}{216}$, $\frac{27}{216}$, $\frac{25}{216}$, $\frac{21}{216}$, $\frac{15}{216}$, $\frac{10}{216}$, $\frac{6}{216}$, $\frac{3}{216}$, $\frac{1}{216}$.

The chance of throwing at least ten is equal to the sum of all these chances$=\frac{135}{216}=\frac{5}{8}$.

The chance that A will win $=\frac{5}{8}$, that he will not win $=\frac{3}{8}$. The chance that B will win $=\frac{5}{8}.\frac{3}{8}$, that he will not win $=(\frac{3}{8})^2$. The chance that C will win $=\frac{5}{8}.(\frac{3}{8})^2$. The chance that A will win the second throw $=\frac{5}{8}.(\frac{3}{8})^3$, that B will win$=\frac{5}{8}.(\frac{3}{8})^4$, that C will win $=\frac{5}{8}.(\frac{3}{8})^5$.

$$\therefore\ A\text{'s chance } =\tfrac{5}{8}+\tfrac{5}{8}(\tfrac{3}{8})^3+\tfrac{5}{8}(\tfrac{3}{8})^6+\ldots\ldots=\tfrac{64}{97},$$

$$B\text{'s chance } =\tfrac{5}{8}(\tfrac{3}{8})+\tfrac{5}{8}(\tfrac{3}{8})^4+\tfrac{5}{8}(\tfrac{3}{8})^7+\ldots\ldots=\tfrac{24}{97},$$

$$C\text{'s chance } =\tfrac{5}{8}(\tfrac{3}{8})^2=\tfrac{5}{8}(\tfrac{3}{8})^5+\tfrac{5}{8}(\tfrac{3}{8})^8+\ldots\ldots=\tfrac{9}{97},$$

$$A\text{'s expectation } =(\tfrac{64}{97}\times30)-10=\$9\tfrac{77}{97},$$

$$B\text{'s expectation } =(\tfrac{24}{67}\times30)-10=-\$2\tfrac{56}{97},$$

$$C\text{'s expectation } =(\tfrac{9}{97}\times30)-10=-\$7\tfrac{21}{97}.$$

Also solved by *L. V. Roy*, *P. H. Philbrick*, *J. F. W. Scheffer*, *H. C. Whitaker*, and *W. H. Draughon*

PROBLEMS.

4. Proposed by G. B. M. ZERR, Principal of High School, Staunton, Virginia.

Four points taken at random in each half, made by the transverse axis, of an ellipse, are joined in such a way by straight lines as to enclose an octagonal surface; find the mean area of this surface.

5. Proposed by DE VOLSON WOOD, M. A., C. E., Professor of Mechanical Engineering, Stevens Institute of Technology, Hoboken, New Jersey.

An actual case suggested the following:

An equal number of white and black balls of equal size are thrown into a rectangular box, what is the probability that there will be contiguous contact of white balls from one end of the box to the opposite end? As a special example, suppose there are 30 ball in the length of the box, 10 in the width, and 5 (or 10) layers deep.

Solutions to these problems should be received on or before May 1st.

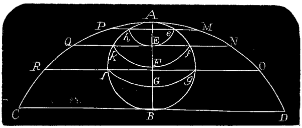
MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

1. Proposed by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

To divide the arc of a cycloid into eight equal parts.

I. Solution by ALFRED HUME, C. E., Professor of Mathematics, University of Mississippi; H. W. DRAUGHON, Clinton, Louisiana; and the Proposer.



Let CAD be the cycloid, AB the diameter of generating circle. Divide AB into four equal parts at E, F, G, and with A as a center and radii equal to AE, AF, AG, draw arcs cutting the circle at e, h; f, k; g, l. Through e, h; f, k; g, l draw parallels to CD. Then, arc $AM=2$ chord $Ae=2\,AE$,

arc $AN=2$ chord $Af=2AF=4AE$,

arc $AO=2AG$, arc $AD=2AB$.

$\therefore AN=2AM,\ AO=3AM,\ AD=4AM.$

$\therefore AM=MN=NO=OD=AP=PQ=QR=RC.$

II. Solution by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy, Pacific College, Santa Rosa, California.

Suppose the curve to be divided into n equal parts, n being any integral number. Through the points of division, which are points symmetrical with respect to the axis, let right lines be drawn. Call the distances severally from the points where these lines intersect the axis to the vertex equal to x.

Let $2r$ = height of the cycloid; then $8r$ = the entire length of the curve. If n be odd, $\sqrt{8rx}=\frac{1}{2n}\cdot 8r,\ \frac{3}{2n}.8r,\ \frac{5}{2n}.8r$, etc.

Whence respectively,

$$(1)\ x=\left(\frac{1}{2n}\right)^2.\,8r,\ \left(\frac{3}{2n}\right)^2.\,8r,\ \left(\frac{5}{2n}\right)^2.8r\ldots\ldots\left(\frac{n-2}{2n}\right)^2.8r=\left(\frac{1}{n}\right)^2 2r,$$
$$\left(\frac{3}{n}\right)^2 2r,\left(\frac{5}{n}\right)^2 2r\ \cdots\cdots\left(\frac{n-2}{2n}\right)^2 2\,r.$$

If n be even, $\sqrt{8rx}=\frac{0}{n},\ \frac{8r}{n},\ \frac{16r}{n},\ \frac{24r}{n}$, etc.

$$\therefore\ (2),x=\left(\frac{0}{n}\right)^2 8r,\ \left(\frac{1}{n}\right)^2 8r,\ \left(\frac{2}{n}\right)^2 8r,\ \left(\frac{3}{n}\right)^2 8r,\ \text{etc.}$$

Making $n=8$, in (2), we have, $x=0,\ \frac{r}{8},\ \frac{r}{2},\ \frac{9}{8}\,r$. If these distances be measured down the axis from the vertex, the ordinates to the points so determined will mark the curve in the points of division required.

Also solved by *P. H. Philbrick, H. C. Whitaker, C. E. Myers*, and *J. H. Beach*.

2. Proposed by SYLVESTER ROBINS, North Branch Depot, New Jersey.

Give the diminsions of thirteen rational trapezoids each one having 1885 for its parallel bisector; and as many more wherein each bisector is 1105.

Solution by A. H. BELL, Hillsboro, Illinois.

The bisector 1885 of the trapezoid, also the bisector 1105, given in the problem, are composed of the product of three prime numbers, of the form of

$4m+1$. All bisectors thus formed, will be the bisectors for 13 trapezoids.

List for 1885, giving the parallel sides.		List for 1105.	
No. 1.	65—1885—2665.	No. 1.	73—1105—1561.
2.	377—1885—2639.	2.	155—1105—1555.
3.	593—1885—2599.	3.	221—1105—1547
4	667—1885—2581.	4.	367—1105—1519.
5.	719—1885—2567.	5.	391—1105—1513
6.	965—1885—2485.	6.	455—1105—1495.
7.	1015—1885—2465.	7.	533—1105—1469.
8.	1085—1885—2385.	8.	595—1105—1445.
9.	1297—1885—2329.	9.	799—1105—1343.
10.	1363—1885—2291.	10.	809—1105—1337.
11.	1409—1885—2263.	11.	923—1105—1261.
12.	1537—1885—2171.	12.	995—1105—1205.
13.	1651—1885—2093.	13.	1057—1105—1151.

Observe that had the conditions, included "prime" then the four trapezoids only answering for 1885, are Nos. 3, 5, 9, and 11. For 1105, Nos. 1, 4, 10, and 13, as given above.

Solved under a slightly different conception by *G. B. M. Zerr*, and exhaustively discussed by Hon. *Josiah H. Drummond*, of Portland, Maine, in a paper which we expect to give later.

3. Proposed by J. A. CALDERHEAD, B. Sc., Superintendent of Schools, Limaville, Ohio.

Given the simultaneous angular velocities of a body about the principal axes through its center of of inertia, find the position of these axes in space at any assigned instant.

I. Solution by the Proposer.

Represent the axes, at first by α, β, γ; and if q be the quaternion defined in § 372 (*Tait's Quaternions*), and ω_1, ω_2, ω_3 (functions of the time) represent the angular velocities about the three axes in their new positions, we have obviously

$$2\,Vqq^{-1}=(e=)q(\omega_1\alpha+\omega_2\beta+\omega_3\gamma)q^{-1}.$$

Integrating this gives q, and the axes are then $q\alpha q^{-1}$, $q\beta q^{-1}$, $q\gamma q^{-1}$.

II. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Angular velocities are resolved and compounded as are linear velocities. If ω_1, ω_2, ω_3 be the angular velocities designated in the problem, the resultant angular velocity is $\sqrt{\omega_1^2+\omega_2^2+\omega_3^2}=\omega$.

The direction cosines are $\frac{\omega_1}{\omega}$, $\frac{\omega_2}{\omega}$, $\frac{\omega_3}{\omega}$, determining the required positions.

PROBLEMS.

8. Proposed by H. C. WHITAKER, B. S., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

Find a general expression for the (integral) co-ordinates of a triangle with sides of integral lengths.

9. Proposed by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Wires of five different metals A,B,C,D,E, having resistances a,b,c,d,e, have their ends soldered together at two junctions which are maintained at different constant temperatures. If the strength of current in E, when all five wires are continuous, is S, the strength of current when B, C, D, are cut is S_a, the strength of current when A, C, D, are cut is S_b, the strength of current when A, B, D, are cut is S_c, find the strength of current S_x, when A, B, C, are cut.

Solutions to these problems should be received on or before May 1st.

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

NOTE TO ARTICLE ON PAGE 42. On the hypothesis that at a given point in a given straight an angle can be made equal to any angle of a triangle, it follows, in Lobatschewsky's geometry, that an angle can be made less than any one given finite angle however small. But this need not surprise anyone, since in Euclid's geometry, we may get two angles each half of any given finite angle however small, by simply bisecting the given angle.—G. B. ALSTED.

I. Is theorem 4 of Lobatschewsky's theory of parallels sound? It reads as follows: "Two straight lines perpendicular to a third never intersect, how far soever they be produced."

II. Is Lobatschewsky's theorem 4 in harmony with the assumtion that two straight lines perpendicular to a third do intersect "at infinity?"

III. Is Lobatschewsky's theorem 4 in harmony with the assumtion that two straight lines perpendicular to a third do intersect "at a finite distance?"

IV. In order that a straight line may be finite must it have a beginning and a termination, that is, two ends?

V. Can a straight line having two ends be infinite in length?

VI. In his theorem 16, Lobatschewsky says: "All straight lines which in a plane go out from a point can, with reference to a given straight line in the same plane, be divided into two classes—into *cutting* and *not-cutting*.

The *boundary lines* of the one and the other class of those lines will be called *parallel to the given line*." Does Lobatschewsky regard the boundary line as a *cutting* or a *not-cutting* line, or neither?

In proposition 33 he seems to teach that the boundary line between the cutting and the non-cutting lines is a cutting line. For he says, "hence not only does the distance between two parallels decrease (Theorem 24), but with the prolongation of the parallels towards the side of the parallelism this at last wholly vanishes."

VII. What is Lobatschewsky's reason for adopting the hypothesis that the boundary line is a *cutting line* rather than a *non-cutting one?*

JOHN N. LYLE,
Westminster College, Fulton, Missouri.

NOTES.

Professor J. W. Nicholson, A. M., LL. D., of Louisana State University, has just finished the manuscript of a Differential and Integral Calculus, on a *new plan* or *doctrine*; it will be published by the University Publishing Company of New York and will be ready for use by September.

The first number of *The Mathematical Gazette*, a new Journal to be published by the Association for the Improvement of Geometrical Teaching and to be edited by Mr. E. M. Langley will contain, besides questions and solutions, articles on the following subjects: (1) Boscovich's treatment of the conic by means of the eccentric angle; (2) Herbart's view of the place of mathematics in education; (3) Greek Geometers antecedent to Euclid; (4) Arithmetical approximation.

It will interest the readers of the MONTHLY to know that Professor F. I. Stewart has asked permission to present to the Canadian educational public an account of our well known contributor, Dr. Halsted's, innovations and contributions in geometry. Also that Nicholas Murry Butler, editor of the Educational Review has requested Dr. Halsted to prepare an article of one thousand words on Cajori's History of Mathematics.

Our valued contributors, Professors J. K. Ellwood, A. M., and J. R. Baldwin, A. M., were elected members of the New York Mathematical Society at a regular meeting held in New York City, March 3d.

M. Eugene Catalan, the famous geometry and professor of higher analysis in the University of Liege, Belgium, died at Liege on the 14th of February.

Dr. Halsted received from Monseur le Secretaire Roffy the official notification that he was at the seance of the 7th of March (this month) chosen membre de la Societe Mathematique de France. Dr. Halsted is the fourth American so honored. We congratulate Dr. Halsted and feel that the society to which he was chosen not only conferred an honor on Dr. Halsted, but also honored itself by choosing so able a man.

EDITORIALS.

This number of the MONTHLY was mailed the 3rd of April, which was due to the fact that when about through printing, the publishers broke their press. Hereafter we will try to mail the MONTHLY between the 20th and 25th of each month.

ALL who have not the January and February Numbers will please notify us at once.

PROFESSOR ROBERT J. ALEY, of the Indiana State University, is still adding to his list of subscribers. We are very greateful to our friends for the efforts they have put forth in endeavoring to extend the circulation of the MONTHLY. It ought to have a circulation of several thousand before the end of the year.

ALL subscriptions should be sent to B. F. Finkel, Kidder, Missouri.

A number of errors have crept into this issue. These are no less annoying to the editors than to our subscribers.

BOOKS AND PERIODICALS.

Mathematics for Common Schools. A Graded Course in Arithmetic, with Simple problems in Algebra and Geometry, by John H. Welsh, Associate Superintendent of Public Instruction, Brooklyn, New York. 8vo, 800 pp. Boston: D. C. Heath & Co.

This work is issued in three parts, bound separately or together.

Part I. An Elementary Arithmetic for the upper primary grades, includes *simple* work in denominate numbers, common and decimal fractions, and measurements.

Part II. An Intermediate Arithmetic, for lower grammar grades, reviews the work of the Elementary Arithmetic, and takes up the easier (and more important) parts of percentage and interest. A chapter on easy equations of one unknown quantity, which may be commenced at any time throughout the course, is appended.

Part III. A Higher Arithmetic, for the upper grammar grades, extends the work in percentage and interest, with the various applications, using the algebraic methods for the solution of the so-called "problems."

A very commendable feature of this work is the introduction of the simple algebraic methods, thus making the pupil familiar with many of the algebraic operations before he takes it up as a distinct subject.

Throughout the three parts are developed the ordinary applications of geometry to finding the areas, surfaces, and volumes of geometrical magnitudes.

The books are handsomely bound in cloth and leather, and the typography is first class. For terms for introduction write the publishers, who will be glad to answer any inquiries respecting this work. B. F. F.

The Non-Euclidean Geometry. Geometrical Researches on the Theory of Parallels, by Nicholaus Lobatschewsky. Translated from the original by *George Bruce Halsted*, A.M.,Ph. D., ex-Fellow of Princeton College and Johns Hopkins University, Professor of Mathematics in the University of Texas, Austin, Texas. [Fourth Edition, 1892.] Price, bound and post paid, $100.

No one should be consider a teacher of geometry who has not read and studied the difficulties attending the development of a consistent theory on parallels. The theory here presented is logical, consistent, and mathematical.

What the result of further study and investigation in Non-Euclidean geometry will be no one may say: but that it will be far reaching and have great bearing upon subsequent scientific development can not be doubted.

Great credit is due Dr. Halsted for his lucid translation of this important theory. B. F. F.

On the Definitions of the Trigonometric Functions. By Alexander Macfarlane, M. A., D. Sc., LL. D., Professor of Physics in the University of Texas.

This valuable paper by Professor Macfarlane was read before the Mathematical Congress at Chicago, August 22, 1893, and is one of the series of able papers he has written on Space Analysis. In the present paper, the author first reviews critically the historical definitions of the trigonometric terms, and the definitions, triangular, circular, or hyperbolic, given in the best modern treatices, and then proceeds to devise a logical system of definitions which will apply to space-analysis and that modern trigonometry, which includes the properties both of circular and hyperbolic functions, and will bring within the same domain the properties of the elliptic, general hyperbolic,

and other functions. In the above pamphlet attention is mostly given to trigonometry in a plane, trigonometry in space being reserved for a separate paper. Copies of this instructive pamphlet (50 pages) can be had of the author, University of Texas. Price, 50 cents.

J. M. C.

Academic Geometry. By William F. Bradbury, A. M., Head Master of the Cambridge Latin School, 1893. 8 vo, 384pp. Price $1.25. Boston: Thompson, Brown, and Company.

This book in the "Bradbury's Eaton's Mathematical Series" has many good features. The definitions are clear and accurate, and in the demonstrations the successive logical steps are concisely stated. The exercises for original demonstration, are numerous and so arranged as to be progressive in order of difficulty. Many practical problems are given throghout the book. The author introduces the terms "Normal to a Plane," and "Aspect of a Plane" and thus extends to planes the same idea that is used in the definition and treatment of lines and angles in the first book. J. M. C.

Public School Algebra on the inductive method. By C. Clarkson, B. A., Principal of Seaport Collegiate Institute and formerly Principal of the Provincial Model School, Toronto. 188p. Toronto: The W. J. Gage Co.

This book is intended as an Introductory Series of Developement Lessons to form a guide to oral teaching and a thorough introduction to larger works. The guiding principles of the book are to follow the line of least resistance, to seek practical applications from the commencment, and to postpone all maters that are abstruse to a second and more advanced course. As an aid to oral teaching the book will prove very helpful.

Two papers: (1) *A Chapter on the Digets;* (2) *A Chapter on the Property of Numbers—Curios in Mathematics*, by S. C. Gould. Reprinted from *Notes and Queries*, 1892.

The nine digits are susceptible of many singular and peculiar evolutions, some of which are brought together in this paper as a sort of digest. Many of the peculiar properties of numbers are also here collected and accredited to their discovers as far as known. The pamphlet contains 32 pages and is brimful of the curious in mathematics.

J. M. C.

The Mathematical Magazine; A Journal of Elementary and Higher Mathematics. Edited and Published by Artemas Martin, M. A., Ph. D., LL.D., Washington, D. C. Issued Quarterly. Price, $1.00 per year.

We have just received advanced sheets of the October Number, 1893, of this valuable journal, and are delighted with its contents. On pages 129--134 is a solution to the "Four-Pennies Problem" by our valued contributor, Professor Henry Heaton, M. S., Atlantic, Iowa. The integration in the solution was not performed as the practical importance of the solution would not warrant the great amount of labor necessary to perform the integration.

The partial solution is masterly and is illustrated by five beautiful diagrams.

On page 135 is a new and Expeditious Method of Computing the Square Root, by Charles H. Kummel. Other papers are as follows:
On Rational Triangles, by C. A. Roberts and About Square Numbers Whose Sum Is A Square Number—III., by Dr. Artemas Martin. Five solutions to problem 121 are published, it being the only problem solved. Ten very interesting problems are proposed for solution. We hope that the October Number, 1893, of this very excellent Journal may find many anxious readers.

B. F. F.

The Educational Times, London, England, for March contains, besides the usual amount of valuable matter, the solution of 14 excellent problems and 36 new problems proposed. One is sure to find in each issue of this periodical many elegant solutions to difficult problems.

Bibliography on the Polemic Problem: What is the Value of π. Compiled by S. C. Gould, Editor of "Notes and Queries," 1888, Manchester, N. H.

The compiler states that of the 100 titles given in this bibliography, 52 are bound volumes, 32 are pamphlets, 7 are broadsides, and the remaining are communications to the press. The results of 63 of these writers are given, tabulated and classified, for comparison. This Bibliography of Cyclometry and Quadratures gives more than thirty publications by American writers who believe they have found the true and exact ratio. The above is a neat pamphlet of 32 pages, and gives a brief review of most of the works mentioned therein. J. M. C.

The School Visitor, Devoted to Practical Mathematics, Examination Work, Notes, Queries, and Answers. Edited and Published by John S. Royer, Superintendent of Schools, Versailles, Ohio. Published monthly. Price, $1.00 per year.

This progressive journal is just what teachers and students want. In the Feb. No. are published the solutions to six poblems and the solution to a special problem Seventeen problems are proposed for solution. Nine pages are devoted to practical examination work, five pages to answers to Queries, and one page to new Queries.

B. F. F.